Module 5: Projections

Suppose that $E$ is a vector space with dot product $< , >$ and that $\left\{ \Theta_p \right\}_{p=1}^{n}$ is an orthogonal sequence.

Let $S$ be the span of the $\Theta_p$'s.

That is,

$$S = \left\{ u : u = \sum_{p=1}^{n} \alpha_p \Theta_p , \alpha_p \text{ real} \right\}$$
If $f$ is in $E$ then the Fourier Expansion

$$\sum_{p=1}^{n} \frac{\langle f, \Theta_p \rangle}{\langle \Theta_p, \Theta_p \rangle} \Theta_p$$

is in $S$ and is closer to $f$ than any other element in $S$. 
Choosing the closest point in $S$ generates a projection: $P^2 = P$.

That is: Take $f$. Form the Fourier Expansion of $f$. Call what you get $P(f)$. Form the Fourier Expansion of $P(f)$. What you get with this second expansion is $P(f)$ again.
Problem: Obtain the projection onto a plane through the origin.
Solution: Get two linearly independent vectors in the plane: $v_1$ and $v_2$.

Perform the Gramm Schmidt process on these two vectors to construct $\Theta_1$ and $\Theta_2$.

Construct the Fourier Expansion for any $u$:

$$P(U) = \frac{\langle u, \Theta_1 \rangle}{\langle \Theta_1, \Theta_1 \rangle} \Theta_1 + \frac{\langle u, \Theta_2 \rangle}{\langle \Theta_2, \Theta_2 \rangle} \Theta_2.$$
Example:

\[ V1 = [1,1,1] \text{ and } v2 = [1,0,1]. \ U = [1,2,2]. \]

\[ \Theta_1 = [1,1,1] \text{ and } \Theta_2 = [1/3,-2/3,1/3] \]

\[ P(u) = a_1 \Theta_1 + a_2 \Theta_2 = \frac{5}{3} \Theta_1 + \left(-\frac{1}{2}\right) \Theta_2 = \left[\frac{3}{2}, 2, \frac{3}{2}\right] \]
Example: Let 
\[ F(x) = \begin{cases} 
0 & \text{if } x < 1/2 \\
1 & \text{if } x > 1/2
\end{cases}. \]

Get the best approximation for \( f \) on \([0,1]\) using 
\( \{ \sin(x), \sin(2\pi x), \sin(3\pi x), \sin(4\pi x) \} \).

These sine terms are already orthogonal in \( C([0,1]) \). We simply compute the coefficients.

\[
a_n = \frac{\int_0^1 f(x) \sin(n\pi x) \, dx}{\int_0^1 \sin(n\pi x)^2 \, dx}
\]
Graph of Heaviside(x-1/2) and
\[ \frac{2}{\pi} \sin(\pi x) - \frac{2}{\pi} \sin(2 \pi x) + \frac{2}{(3 \pi)} \sin(3 \pi x) \]
Assignment: See the Maple Worksheet

In this Module 5 we have
1. said what we mean by a projection,
2. explained that the Fourier Expansion with orthogonal vectors is a projection,
3. projected onto a plane, and
4. obtained and graphed the Fourier Expansion for a function.