Module 9: Convergence for Fourier Series

Three general types of convergence in $C([0, 1])$: normed, pointwise, and uniform.

We apply these ideas to the special case that we have a function $f$ and its associated Fourier series:

$$S_n(x) = \sum_{p=0}^{n} \langle f, \varphi_p \rangle \varphi_p(x)$$
Definitions:
A function is sectionally continuous on an interval \([a, b]\) if it is continuous on that interval except for possibly a finite number of jumps and removable discontinuities.

A function is sectionally smooth on an interval \([a, b]\) if \(f\) and \(f'\) are sectionally continuous on the interval \([a, b]\).
Examples: The function \( f(x) = \text{signum}(x) \) is sectionally continuous on \([-1, 1]\), but the function \( g(x) = 1/x \) is not sectionally continuous on that interval.

Example: The function \( \sqrt{|x|} \) is (even) continuous, but not sectionally smooth on \([-1, 1]\) because the derivative goes to infinity as \( x \) approaches zero.
THEOREM: If the function $f$ is sectionally smooth and periodic with period $2c$, then at each point $x$ the Fourier series for $f$ converges to

$$\left[ f(x^+) + f(x^-) \right] / 2.$$

Example: The function $\text{signum}(x)$ is sectionally smooth. Therefore the Fourier series for this function converges

to $1$ for $0 < x < 1$,

to $-1$ for $-1 < x < 0$, and

to $0$ for $x = -1, 0, \text{ or } 1$. 
Fourier approximation for signum(x).
THEOREM: If the series \( \sum_{n} |a_n| + |b_n| \) converges, then the Fourier series for \( f \) converges uniformly in the interval \([-c, c]\).

Example: Take the series with the b's zero and
\[ a_n = \frac{4}{n^2 \pi} \] for \( n \) even and zero of \( n \) odd.
THEOREM: If $f(x)$ is periodic, continuous, and has a sectionally continuous derivative, then the Fourier Series corresponding to $f$ converges uniformly to $f(x)$ for the entire real line.

Take $f(x) = |x|$ on the interval $[-\pi, \pi]$. It is continuous and its periodic extension is continuous. The derivative is sectionally continuous. The coefficients are the ones used in the previous example.
Fourier Approximation for $|x|$.
The function \( f(x) = x \) on the interval \([-1, 1]\) is continuous there. The periodic extension is sectionally continuous, but not continuous. The Fourier Series does not converge uniformly.
Fourier Approximation for $x$ on $[-1, 1]$. 
Assignment: See the Maple Worksheet

In this ninth module we have:
Contrasted the three types of convergence for Fourier Series and provided examples. There were three important Theorems which gave conditions to imply the nature of convergence.