Module 13: Review of Elementary Differential Equations II

Question 1. Suppose \( \lambda > 0 \). Which of these is a pair of linearly independent solutions for \( Y'' - \lambda^2 Y = 0 \) on \([0, \pi]\)?

a. \( \exp(\lambda x) \) and \( \exp(-\lambda x) \),

b. \( \sin(\lambda x) \) and \( \cos(\lambda x) \),

c. \( \sinh(\lambda x) \) and \( \cosh(\lambda x) \),

d. \( \sinh(\lambda x) \) and \( \sinh(\lambda (\pi - x)) \)
Question 2. Suppose $\lambda > 0$. Which of these is a pair of linearly independent solutions for $Y'' + \lambda^2 Y = 0$ on $[0, \pi]$?

- e. $\exp(\lambda x)$ and $\exp(-\lambda x)$,
- f. $\sin(\lambda x)$ and $\cos(\lambda x)$,
- g. $\sinh(\lambda x)$ and $\cosh(\lambda x)$,
- d. $\sinh(\lambda x)$ and $\sinh(\lambda (\pi - x))$
Question 3. Suppose $\lambda > 0$. Which of these is a bounded solution for

$$Y'' - \lambda^2 Y = 0 \text{ on } [0, \infty)?$$

a. $\exp(\lambda x)$  

b. $\exp(-\lambda x)$  

c. $\sinh(\lambda x)$  

d. $\cosh(\lambda x)$

There are two issues here: which is a solution and which is bounded on the specified interval.
Question 4. Which of these is a bounded solution on the interval [0, 5] for the differential equation
\[ r^2 R''(r) + r R'(r) - 9 R(r) = 0 \]?

a. \( \exp(3 \, r) \)  
b. \( r^3 \)  
c. \( \sin(3 \, r) \)  
d. \( \exp(-3 \, r) \)  
e. \( 1/r^3 \)  
f. \( \cosh(3 \, r) \)
Question 5. If \( u(x, y) = \)

\[
\sum_{p} a_p \sin(px) \sinh(py) + \sum_{p} b_p \sin(px) \sinh(p(\pi - y))
\]

and

\[
u(x,0) = 0, \quad u(x, \pi) = \sin(2x)
\]

what are the \( a_p \)'s and \( b_p \)'s?

We have two pieces of information.
\[ \sum a_p \sin(px) \sinh(py) + \sum b_p \sin(px) \sinh(p(\pi - y)) \]

\[ u(x,0) = 0, \quad u(x, \pi) = \sin(2x) \]

The first of these implies that all the \( b_p \)'s = 0, and the second implies that all the \( a_p \)'s = 0, except \( a_2 = 1/\sinh(2\pi) \).
Graph of $\sin(2x) \sinh(2y) / \sinh(2\pi)$
Question 6. If \( u(r, \theta) = \) 

\[ \sum_p a_p \sin(p \theta) r^p + \sum_p b_p \cos(p \theta) r^p \]

and

\[ u(1, \theta) = 1 + 3 \cos(3 \theta) + 5 \sin(2 \theta) \]

then what is \( u(r, \theta), \ u(0,0), \ \text{and} \ u(1/2, \pi/4)? \)
\[
\sum_{p} a_p \sin(p \theta) r^p + \sum_{p} b_p \cos(p \theta) r^p
\]

\[u(1, \theta) = 1 + 3 \cos(3 \theta) + 5 \sin(2 \theta)\]

This implies that all \(a_p\)'s = 0 and \(b_p\)'s = 0

except \(b_0 = 1, b_3 = 3,\) and \(a_2 = 5.\)
\[ u(r, \theta) = 1 + 3 \, r^3 \cos(3 \, \theta) + 5 \, r^2 \sin(2 \, \theta) \]

\[ u(0,0) = 1 \quad \text{and} \quad u(1/2, \pi/4) = 1 - 3/8 \sqrt{2} + 5/4. \]
Question 7. What are all the eigenvalues of the self-adjoint, Sturm-Liouville Problem

\[ y'' = \mu y, \text{ with } y(0) = y(1) = 0 \] 

We break the problem into two cases.

First, suppose that \( \mu > 0 \).

Take \( \mu = \lambda^2 \). Thus, we seek numbers \( \lambda \) such that

\[ y'' = \lambda^2 y \text{ with } y(0) = y(1) = 0. \]
\[ y'' = \lambda^2 y \] with \( y(0) = y(1) = 0 \).

has general solutions of the form
\[ y(x) = A \exp(\lambda x) + B \exp(-\lambda x). \]

\[ y(0) = 0 \implies 0 = A + B. \]

\[ y(1) = 0 \implies 0 = A \exp(\lambda) + B \exp(-\lambda). \]

Thus \( A = 0 = B \).
\[ y'' = \mu \, y, \text{ with } y(0) = y(1) = 0? \]

Second case: \( \mu < 0 \). Take \( \mu = -\lambda^2 \).

\[ y'' = -\lambda^2 \, y \text{ with } y(0) = y(1) = 0. \]

General solution is
\[ y(x) = A \sin(\lambda \, x) + B \cos(\lambda \, x). \]

\[ y(0) = 0 \Rightarrow 0 = B. \]
\[ y(1) = 0 \Rightarrow 0 = A \sin(\lambda). \]

\[ 0 = \sin(\lambda) \quad \text{so that} \quad \lambda = n \, \pi \text{ and } \mu = -n^2 \, \pi^2. \]
Eigenvalues are \(- n^2 \pi^2\) and eigenfunctions are \(\sin(n \pi x)\).

Surprised?

Assignment: See Maple Worksheet

In this Module 13 we have
1. Examined bounded solutions for Sturm-Liouville Problems,
2. Found eigenvalues for a Sturm-Liouville Problem, and
3. Identified coefficients of trigonometric series.