Module 15: The Critical Notion:
Complete Orthonormal Sequences

Definition: The collection of vectors
\[ \{ \Theta_1, \Theta_2, \Theta_3, \ldots \} \]
is called **orthogonal** if

\[ \langle \Theta_i, \Theta_j \rangle = 0, \text{ whenever } i \neq j. \]

The family is **orthonormal** if each \[ |\Theta_i| = 1. \]
Here is the important inequality that we derived:

\[ \left\| f - \sum_p a_p \Phi_p \right\|^2 = \|f\|^2 + \sum_p \left( \langle f, \Phi_p \rangle - a_p \right)^2 - \sum_p \|\langle f, \Phi_p \rangle\|^2 \]

Several important facts followed from this inequality.
Fact 1. Suppose that \( n \) is a positive integer, that
\[
\{ \Phi_1, \Phi_2, \Phi_3, \ldots, \Phi_n \}
\]
is an orthonormal sequence, and that
\[
S_n = \text{span of the } \{ \Phi_1, \Phi_2, \Phi_3, \ldots, \Phi_n \}.
\]
If \( f \) is in the space, then the closest element in \( S_n \)
to \( f \) is given by the Fourier expansion:
\[
\sum_{p=1}^{n} \langle f, \Phi_p \rangle \Phi_p
\]
We argue this:
\[
\left\| f - \sum p a_p \Phi_p \right\|^2 = \| f \|^2 + \sum p \left( \langle f, \Phi_p \rangle - a_p \right)^2 - \sum p \| \langle f, \Phi_p \rangle \|^2
\]
Fact 2. If the $\Phi_p$ 's form an infinite sequence, then the infinite series
\[ \sum_p \left| \langle f, \Phi_p \rangle \right|^2 \]
converges.

We argue this:

\[ \left\| f - \sum_p a_p \Phi_p \right\|^2 = \|f\|^2 + \sum_p \left| \langle f, \Phi_p \rangle - a_p \right|^2 - \sum_p \left\| \langle f, \Phi_p \rangle \right\|^2 \]
Fact 3. If the $\Phi_p$'s form an infinite sequence, then the infinite series

$$\sum_{p=1}^{\infty} \langle f, \Phi_p \rangle \Phi_p$$

in the vector space converges.

We argue this: if $n > m$ then

$$\left\| \sum_{p=m}^{n} \langle f, \Phi_p \rangle \Phi_p \right\|^2 = \sum_{p=m}^{n} |\langle f, \Phi_p \rangle|^2$$

and this latter converges.
Definition: An orthogonal sequence is complete if the only vector in the space that is orthogonal to every element in the sequence is the zero vector.

Fact 4. Suppose \( \{ \Theta_1, \Theta_2, \Theta_3, \ldots \} \) is a complete orthonormal sequence. Then

\[
\sum_{p=1}^{\infty} \langle f, \Phi_p \rangle \Phi_p
\]

converges to \( f \) in norm.

We argue this.
We argue that

$$\sum_{p=1}^{\infty} \langle f, \Phi_p \rangle \Phi_p = f$$

by showing that the element $g$ defined as

$$f - \sum_{p=1}^{\infty} \langle f, \Phi_p \rangle \Phi_p$$

is orthogonal to each $\Phi_i$. 
The projection of $f$ onto the span of $\sin(nx)$'s and onto the span of $\sin(mx)$'s where $n$ is odd and $m$ is even.
Assignment: See Maple Worksheets.

In this Module 15, we have provided an argument in support of the proposition that the Fourier Series for a function f, using a complete, orthonormal family, converges in norm to f.