Module 16: The Simple Heat Equation

We consider the simple heat equation:

\[
\frac{du}{dt} = \frac{d^2u}{dx^2}, \quad u(t, 0)=0, \quad u(t, 1) = 0
\]

\[
u(0, x) = f(x).
\]
There is a physical interpretation.

The method of solving this equation is called: Separation of Variables.

\[ u(t,x) = X(x) \ T(t). \]

\[ \frac{du}{dt} = \frac{d^2u}{dx^2} \quad \text{leads to} \]
\[ X \ T' = X'' \ T, \]

or

\[ \frac{T'}{T} = \frac{X''}{X}. \]
\[
\frac{T'}{T} = \frac{X''}{X} \implies \frac{T'}{T} \text{ and } \frac{X''}{X} \text{ are constant.}
\]

\[U(t, 0) = X(0) \quad T(t) = 0 \implies X(0) = 0\]
\[U(t, 1) = X(1) \quad T(t) = 0 \implies X(1) = 0\]

\[X'' = \mu X, \text{ with } X(0) = 0 = X(1).\]

See Lecture 13, Question 7. There:
\[
\mu \text{'s are } -n^2 \pi^2 \text{ and } x \text{'s are } \sin(n \pi x).
\]
What about the possible $T$ solutions? The quotient $T'/T$ is the same quotient, so that

$$T' = - n^2 \pi^2 T.$$ 

Solutions for this equation are

$$T(t) = \exp(- n^2 \pi^2 t).$$

Consequently, for each integer $n$, we have a solution.
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$$u(t, x) = \exp(- n^2 \pi^2 t) \sin(n \pi x).$$

Conclusion:

$$u(t, x) = \sum_n c_n \exp(- n^2 \Pi^2 t) \sin(n \Pi x)$$

The condition $u(0, x) = f(x)$ determines the $c_n$'s.

$$C_n = \frac{1}{2} \int_0^1 f(x) \sin(n \Pi x) \, dx.$$
Graph of $f(x) = x(1-x)^3$ and a Fourier approximation. Type convergence?
Graph of $u(t, x)$ with $u(0, x) = x (1-x)^3$
Trace the highpoint:
Assignment: See the Maple worksheet.

In this Module 16, we have solved a simple heat equation with zero boundary conditions and with an initial distribution.