Module 19: Convection Across Boundaries

PDE: \( \frac{dw}{dt} = \frac{d^2w}{dx^2} \),

Boundary Conditions:
\[
\frac{dw}{dx}(t, 0) = 0 \quad \text{and} \quad \frac{dw}{dx}(t,1) = -(w(t,1) - 2).
\]

The Initial Condition: \( w(0, x) = 4x(1-x) + 2 \).

This PDE has a physical interpretation.
The PDE was homogeneous, but the boundary conditions were not.

PDE: \( \frac{du}{dt} = \frac{d^2u}{dx^2} , \)

Homogeneous Boundary Conditions:
\( \frac{du}{dx}(t, 0) = 0 \) and \( \frac{du}{dx}(t,1) = - u(t,1) . \)

Is it clear that the steady state solution is \( v(x) = 2 \) and that \( u = w - 2 \)?

Our Job: find the general solution for the PDE with homogeneous boundary conditions.
Separation of variables leads to two ordinary differential equations, one having boundary conditions.

\[ X'' = -\lambda^2 X, \quad X'(0) = 0, \quad X'(1) + X(1) = 0. \]

\[ T' = -\lambda^2 T \]

\[ X(x) = A \sin(\lambda x) + B \cos(\lambda x). \]

The first boundary condition implies \( A = 0. \)
\[ X(x) = 0 \sin(\lambda \, x) + B \cos(\lambda \, x). \]

\[
0 = X'(1) + X(1) = -B \lambda \sin(\lambda) + B \cos(\lambda)
\]

or

\[
\tan(\lambda) = \frac{1}{\lambda}.
\]

We'll find these \( \lambda \)'s. Then, Eigenvalues are \( \mu = -\lambda^2 \) and eigenfunctions are \( \cos(\lambda \, x) \).
We find a sequence of $\lambda_n$'s and make

$$u(t, x) = \sum c_n \exp(-\lambda_n^2 t) \cos(\lambda_n x).$$

Good luck at trying to get the $\lambda_n$'s in closed form! Rather, we need a numerical procedure.

Recall Newton's Method:

To find $s$ so that $g(s) = 0$, make a first guess and iterate: $x_0 =$ first guess,

$$X_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}.$$. 
Graph of $\tan(x) - \frac{1}{x}$
\[ \lambda_1 = 0.860335890 \]
\[ \lambda_2 = 3.425618459 \]
\[ \lambda_3 = 6.437298179 \]
\[ \lambda_4 = 9.529334405 \]
\[ \ldots \]

\[ u(t, x) = \sum c_n \exp(-\lambda_n^2 t) \cos(\lambda_n x) \cdot \]

\[ c_n = \frac{\int_0^1 (f(x) - 2) \cos(\lambda_n x) \, dx}{\int_0^1 \cos(\lambda_n x)^2 \, dx} \]
Graph of $w(t, x)$
Snapshots of $w(t, x)$ as $t$ increases.
Left end temperature as \( t \) increases
Assignment: See Maple worksheet

In this Module 19, we have investigated the heat equation with convection boundary conditions.