Module 25: A string in a viscous medium

In this worksheet, we suppose that a string is subject to gravity and embedded in a viscous medium.

The problem: find $w$ to satisfy

$$w_{xx} - k \, w_t - g = w_{tt}$$

$$w(t, 0) = 0 = w(t, L)$$

$$w(0, x) = F(x) \quad \text{and} \quad w_t(0, x) = G(x).$$
Simplification: \( v \) and \( w \) related by

\[ v(t, x) = w(t, x) + g x (L-x)/2 \]

or

\[ w(t, x) = v(t, x) - g x (L-x)/2 \]

\[ v_{xx} - k v_t = v_{tt} \]

\[ v(t, 0) = 0 = v(t, L) \]

\[ v(0, x) = F(x) + g x (L-x)/2 \]

\[ v_t (0, x) = G(x). \]
Simplification: \( u \) and \( v \) related by

\[
\begin{align*}
    u(t, x) &= \exp(t \frac{k}{2}) \: v(t, x) \\
    \text{or} \\
    v(t, x) &= \exp(-t \frac{k}{2}) \: u(t, x)
\end{align*}
\]

\[
\begin{align*}
    u_{xx} + k^2 \frac{u}{4} &= u_{tt} \\
    u(t, 0) &= 0 = u(t, L) \\
    u(0, x) &= v(0, x) \\
    u_t(0, x) &= k/2 \: v(0,k) + v_t(0,x)
\end{align*}
\]
Solve for $u$, create $v$, create $w$

\[ w(t, x) = \exp(-k/2 \, t) \, u(t, x) - g \, x \, (L-x)/2 \]

\[ u_{xx} + \frac{k^2}{4} u = u_{tt} \]

\[ u(t, 0) = 0 = u(t, L) \]

With $k$ small, this leads to

\[ X'' = -\alpha^2 \, X, \, X(0) = 0 = X(L) \]

\[ -\alpha^2 = - (\mu^2 + k^2/4) = - \, n^2 \, \pi^2/L \]

\[ T'' = - \mu^2 \, T \]
\[ \mu^2 = n^2 \pi^2 / L - k^2 / 4 \]

**T(t) is \( \sin(\mu \ t) \) or \( \cos(\mu \ t) \)**

**U(t, x) = \[ A_n \sin(\mu \ t) + B_n \cos(\mu \ t) \] \sin(n\pi x / L) **

**Determine coefficients.**

**w(t, x) = \exp(-k/2 \ t) \ u(t, x) - g \ x \ (L-x) / 2**
Example:

\[ w_{xx} - \frac{1}{4} w_t = w_{tt} \]

\[ w(t, 0) = 0 = w(t, \pi) \]

\[ w(0, x) = \sin(2x) \quad \text{and} \quad w_t(0, x) = 0. \]

\[ w(t, x) = \exp(-k/2t) u(t, x) \]
\[ u(t, x) = \sin(2x) \cos(\sqrt{4 - \frac{k^2}{4}} \cdot t) + \frac{k}{2 \sqrt{4 - \frac{k^2}{4}}} \sin(2x) \sin(\sqrt{4 - \frac{k^2}{4}} \cdot t) \]

\[ w(t, x) = \exp(-\frac{k}{2} \cdot t) \cdot u(t, x). \] Initial graph:
Graph of $u(t, x)$
Assignment: See Maple worksheet.

In this Module 25, we have modeled a string in a viscous medium, subject to gravity.