Module 28: Laplace's Equation with Insulated Boundaries

We work the following problem

\[ 0 = u_{xx} + u_{yy} \]

Boundary conditions:

\[ u_y(x,0) = 0, \quad u_y(x,1) = 0 \]
\[ u(0,y) = \sin(\pi y), \quad u_x(1,y) = 0 \]

Not two PDE's for this classroom problem.
\[ Y'' + \lambda^2 Y = 0, \quad Y'(0) = 0 = Y'(1), \]
\[ X'' - \lambda^2 X = 0. \]

**Solutions:**

\[ \lambda_n = n \pi. \]

\[ X_0 = A_0 x + B_0, \quad X_n = \sinh(n \pi x) \text{ or } \sinh(n \pi (1-x)), \]

\[ Y_n(y) = \cos(n \pi y). \]

\[ U(x,y) = A_0 x + B_0 + \sum_n (A_n \sinh(n \pi x) + B_n \sinh(n \pi (1-x)) \cos(n \pi y)) \]
\[
\sin(\pi y) = U(0, y) = B_0 + \sum_{n} B_n \sinh(n\pi)\cos(n\pi y).
\]

\[
B_0 = \int_{0}^{1} \sin(\pi y) \, dy
\]

\[
B_n \sinh(n \pi) = 2 \int_{0}^{1} \sin(\pi y)\cos(n\pi y) \, dy
\]

\[
0 = u_x(1, y) = A_0 + \sum_{n} n\pi(A_n \cosh(n\pi) - B_n)\cos(n\pi y)
\]

\[
A_0=0 \text{ and } A_n = B_n/\cosh(n \pi)
\]
The situation:
Graph of $u$
Contour Lines
We work the following problem
\[ 0 = u_{xx} + u_{yy} \]

Boundary conditions:
\[ u_y(x,0) = 1 - \cos(2 \pi x), \quad u_y(x,1) = 1 \]
\[ u_x(0,y) = 0, \quad u_x(1,y) = 0 \]

That \( u \) is not changing in time has implications for what is mathematically possible here.
We know how to separate variables. The differential equations will be

\[ X'' + \lambda^2 X = 0, \quad X'(0) = 0 = X'(1), \]

and \[ Y'' = \lambda^2 Y. \]

Solutions for this will be

\[ X(x) = 1 \quad \text{or} \quad \cos(n \pi x) \quad \text{if } n > 0 \]

and \[ Y(y) = 1 \quad \text{and} \quad y \quad \text{or} \quad \cosh(n \pi y) \quad \text{and} \quad \cosh(n \pi (1-y)) \quad \text{if } n > 0. \]
\[ X(x) = 1 \quad \text{or} \quad \cos(n \pi x) \text{ if } n > 0 \]

and \[ Y(y) = 1 \text{ and } y \]
or \[ \cosh(n \pi y) \quad \text{and} \quad \cosh(n \pi (1-y)) \text{ if } n > 0. \]

We can construct a general solution.

\[ U(x, y) = A_0 + B_0 y + \]

\[ \sum_n [A_n \cosh(n \Pi y) + B_n \cosh(n \Pi (1-y))] \cos(n \Pi x) \]

We have only to find the A 's and B 's.
Coming in from the bottom is

\[ f(x) = 1 - \cos(2 \pi x) = u_y(x, 0) \]

\[ = B_0 - \sum_{n} B_n n \Pi \sinh(n \Pi) \cos(n \Pi x) \]

Going out the top is

\[ 1 = u_y(x, 1) = B_0 - \sum_{n} A_n \sinh(n \Pi) \cos(n \Pi x) \]

A job for Fourier Series?
Graph of $u$
Assignment: See the Maple worksheet

In this Module 28 we have worked problems which had Neumann boundary conditions.