Module 35: Warm Spheres

An important coordinate system is that of a sphere. In that coordinate system we have

\[ \rho \] is the distance from the origin,

\[ \Theta \] is the angle in the x-y plane; that is, it measures longitude.

\[ \Phi \] is the angle from the top; that is, it measures latitude.
Why is $\rho = 1$, $0 < \Theta < 2\pi$, $0 < \Phi < \pi$ a sphere with radius 1? Recall the connection.

The connection with rectangular coordinates is

\[ x = \rho \sin(\Phi) \cos(\Theta) \]
\[ y = \rho \sin(\Phi) \sin(\Theta) \]
\[ z = \rho \cos(\Phi) \]

That the surface $\rho = 1$ is a sphere follows from

\[ x^2 + y^2 + z^2 = 1. \]
Also, $\rho = 1$, $\Theta = \pi / 4$; $0 < \Phi < \pi$ is a half circle running from the north pole to the south pole of the sphere.

Finally, $\rho = 1$, $\Phi = 49$ degrees, defines a part of the boundary between Western Canada and Western United States.
In this coordinate system, the Laplacian Operator is

\[
\frac{1}{\rho^2} \left\{ \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial u}{\partial \rho} \right) + \frac{1}{\sin(\phi)} \frac{\partial}{\partial \phi} \left( \sin(\phi) \frac{\partial u}{\partial \phi} \right) \\
+ \frac{1}{\sin(\phi)^2} \frac{\partial^2 u}{\partial \theta^2} \right\}
\]

We solve \( 0 = \Delta u, \ u(1, \Phi) = f(\Phi), \) where \( u \) is independent of \( \Theta \).
From the assumption that
\[ u(\rho, \phi) = R(\rho) \Phi(\phi), \]
we are led to this separation of variables situation:

\[ \left( \rho^2 R'(\rho) \right)' / R + \left( \sin(\phi) \Phi' \right)' / \sin(\phi) \Phi = 0. \]

This suggests the ordinary differential equations

\[ \left( \rho^2 R'(\rho) \right)' - \mu^2 R(\rho) = 0, \ 0 < \rho < 1, \]
\[ (\sin(\phi) \Phi')' + \mu^2 \sin(\phi) \Phi = 0, \ 0 < \phi < \pi. \]
\( (\rho^2 R'((\rho)))' - \mu^2 R(\rho) = 0, \ 0 < \rho < 1, \)

\[ (\sin(\phi)\Phi')' + \mu^2 \sin(\phi) \Phi = 0, \ 0 < \phi < \pi. \]

Neither equation has a boundary condition. We have conditions of boundness. In the second equation, we take \( x = \cos(\phi), \ y(x) = \Phi(\phi). \)

Hence,
\[ (\sin(\phi)\Phi')' = \sin(\phi)^3 y'''(x) - 2 \sin(\phi) \cos(\phi) y'(x). \]

The second differential equation above becomes
\[ \sin(\phi)^2 y'''(x) - 2 \cos(\phi) y'(x) + \mu^2 y(x) = 0. \]
In terms of $x$ alone,

\[ \sin(\phi)^2 y''(x) - 2 \cos(\phi) y'(x) + \mu^2 y(x) = 0 \]

becomes

\[ (1-x^2) y''(x) - 2 x y'(x) + \mu^2 y(x) = 0, \quad -1 < x < 1. \]

We are led to consider Legendre Polynomials. We digress to recall these functions.
A recollection of Legendre Polynomials

Here are three ways to conceive of the Legendre Polynomials -- four, if we include Maple.

Method 1: solve the differential equation

\[(1-x^2) y''(x) - 2 x y '(x) + \mu^2 y(x) = 0, \; -1 < x < 1.\]

Take \(\mu^2 = n \; (n+1).\)
Here are graphs of 4 Legendre Polynomials
Method 2: We could apply the Gramm Schmidt Process to 1, x, x^2, ... . We examined these ideas earlier.

Recall that

\[ \int_{-1}^{1} P_n(x) P_m(x) \, dx = 0 \text{ if } n \text{ is not } m. \]
Method 3. We could generate these by taking the appropriate derivatives.

We show that
\[ \frac{1}{2^n n!} d^n (x^2 - 1)^n / dx^n \]

is the \( n^{th} \) polynomial.
Method 4: Use the recursion formulas
\[(n+1) \, P_{n+1}(x) + n \, P_{n-1}(x) = (2 \, n + 1) \, P_n(x) \, x.\]

With this method, we assume you know the first two.
\[P_0(x) = 1, \quad P_1(x) = x,\]
\[2 \, P_2(x) + 1 \, P_0(x) = 3 \, P_1(x) \, x, \quad \text{or}\]
\[2 \, P_2(x) + 1 = 3 \, x^2.\]
Observation 1: if $0 < m < n$, then

$$\int_{-1}^{1} x^m P_n(x) \, dx = 0.$$ 

Observation 2: We have a formula for the norm of $P_n(x)$:

$$\int_{-1}^{1} P_n(x)^2 \, dx = \frac{2}{2n + 1}.$$
Observation 3: We can make polynomial approximations for functions on \([-1, 1]\).
Recall where we were. The partial differential equation led to two ordinary differential equations.

The second differential equation

$$\sin(\phi)^2 y''(x) - 2 \cos(\phi) y'(x) + \mu^2 y(x) = 0$$

became

$$(1-x^2) y''(x) - 2 x y'(x) + \mu^2 y(x) = 0, \quad -1 < x < 1.$$
This led to

\[(1-x^2)y''(x) - 2x y'(x) + n(n+1)y(x) = 0,\]

and Legendre Polynomials.

Here is the first of the differential equations from above, with this \( \mu^2 = n(n+1): \)

\[\left( \rho^2 R'(\rho) \right)' - n(n+1)R(\rho) = 0, \quad 0 < \rho < 1,\]

\[\rho^2 R'' + 2 \rho R' - n(n+1)R(\rho) = 0, \quad 0 < \rho < 1.\]
\[ \rho^2 R'' + 2 \rho R' - n (n+1) R(\rho) = 0, \quad 0 < \rho < 1. \]

has bounded solution \( r^n \).

First, we verify that sums of products of solutions are solutions.

\[
 u(r, \phi) = \sum_n a_n r^n P_n (\cos(\phi))
\]

is a general solution for

\[
 u(r, \phi) = \Delta u
\]
We are now ready to compute the coefficients for a boundary condition. We solve

\[ 0 = \Delta u \quad \text{with} \quad 0 < \rho < 1, \quad 0 < \phi < \pi, \quad 0 < \theta < 2\pi \]

with boundary condition \( u(1, \phi) = f(\phi) \).

The coefficients will be

\[ a_n = \frac{(2n+1)}{2} \int_{-1}^{1} f(\phi) P_n(\cos(\phi)) \, d\phi. \]
Let's take the special case that $f(\phi) = \cos(\phi)$. In this case only $a_1$ is not zero.

Thus, the solution for this problem is

$$u(r, \phi) = r \cos(\phi).$$

How can we illustrate what we have?

(1) Each cross sectional plane parallel to the x-y plane has value $z$. 
(2) Hold $r$ fixed and let $\phi$ go from 0 to $\pi$. See the graph.
Assignment: See the Maple worksheet.

In this Module 35, we have solve the Laplaces equation on a sphere. To do this, we had to rewrite the Laplacian operator in spherical coordinates. Use was made of Legendre Polynomials.