1 Introduction

Let $T_r(n)$ be formed by partitioning the set of $n$ vertices into $r$ parts of equal or nearly equal size, and connecting two vertices by an edge whenever they belong to two different parts. Let $t_r(n) = |E(T_r(n))|$. Let $F$ be a graph and let $ex(n, F)$ denote the maximum number of edges in a graph with $n$ vertices not containing any copy of the graph $F$.

We can now state two well-known theorems in extremal graph theory. The first is Turán’s Theorem [3].

**Theorem 1.1 (Turán)** For $r,n \geq 2$, any $n$-vertex graph $G$ that does not contain $K_{r+1}$ and has $ex(n, K_{r+1})$ edges is isomorphic to $T_r(n)$.

The next theorem is a consequence of the Erdös-Stone theorem, see Diestel [5].

**Theorem 1.2** For every graph $H$ with at least one edge, $ex(n, H) = \frac{\chi(H) - 2}{\chi(H) - 1} \binom{n}{2} + o(n^2)$.

When $\chi(H) = 2$, Theorem 1.2 does not give us much meaningful information. We call it the degenerate case.

The problem of determining the order of magnitude of $ex_2(n, G)$ is notoriously difficult when $G$ is a bipartite graph. Even for such small graphs as $G = C_3$, $G = Q_3$ and $G = K_{4,4}$, the order of magnitude of $ex_2(n, G)$ is not known. Zarankiewicz [19] conjectured that $ex_2(n, K_{s,t}) = \Theta(n^{2-1/s})$ whenever $t \geq s \geq 2$, and this conjecture remains open in general. Kövari, Sós and Turán gave the following general upper bound for $ex_2(n, K_{s,t})$.

**Theorem 1.3** [14] For all $t \geq s \geq 2$, $ex_2(n, K_{s,t}) \leq \frac{1}{2} [(t-1)^{1/s} n^{2-1/s} + (s-1)n]$.

When $t > (s-1)! \geq 1$, the norm graph constructions of Alon, Rónyai and Szabó [2] verity the Zarankiewicz conjecture in those cases.

The $r$-expansion $G^+$ of a graph $G$ is the $r$-uniform hypergraph obtained from $G$ by enlarging each edge of $G$ with a vertex subset of size $r-2$ disjoint from $V(G)$ such that distinct edges are enlarged by disjoint subsets. Let $ex_r(n, F)$ denote the maximum number of edges in an $r$-uniform hypergraph with $n$ vertices not containing any copy of the $r$-uniform hypergraph $F$.

Many problems in extremal set theory ask for the determination of $ex_r(n, G^+)$ for various graphs $G$. Let $t_r(n,l)$ be the size of the Turán $r$-graph $T_r(n,l)$, namely the complete $l$-partite $r$-graph on $n$ vertices in which no two parts differ in size by more than one. Thus the part sizes
are \( n_i = \lfloor \frac{n+i-1}{l} \rfloor \) for \( i \in [l] \). Among all \( l \)-partite \( r \)-graphs on \( n \) vertices, \( T_r(n,l) \) has the most edges. The number of edges in \( T_r(n,l) \) is

\[
t_r(n,l) = \sum_{S \in \binom{[n]}{r}} \prod_{i \in S} n_i \sim \frac{l(l-1) \cdots (l-r+1)}{l^r} \binom{n}{r}
\]

for fixed \( l \geq r \).

Mubayi [15] proved that if \( l \geq r \) is fixed, then \( ex_r(n, K_{l+1}^+) < t_r(n,l) + o(n^r) \) and conjectured that this could be further improved to an exact result for \( n \) sufficiently large. This was subsequently proved by Pikhurko [17].

**Theorem 1.4** [15, 17] Fix \( l \geq r \geq 2 \) and let \( n \) be sufficiently large. Then \( ex_r(n, K_{l+1}^+) = t_r(n,l) \). Moreover equality is achieved only by the Turán \( r \)-graph \( T_r(n,l) \).

Alon and Pikhurko observed that Theorem 1.4 can be extended to any color critical graph \( G \) with \( \chi(G) > r \).

**Theorem 1.5** [1] Fix integers \( l \geq r \geq 2 \) and a graph \( G \) with \( \chi(G) = l + 1 \). Then \( ex_r(n, G^+) \sim t_r(n,l) \).

Theorem 1.5 determines the asymptotics of \( ex_r(n, G^+) \) for all \( G \) with \( \chi(G) > r \), which is the non-degenerate case. Most of the recent activity around \( ex_r(n, G^+) \) has focused on the degenerate case, i.e., when \( \chi(G) \leq r \).

## 2 Some Results

Many well-known results can be stated in terms of \( ex_r(n, G^+) \) for very simple graphs \( G \), such as matchings, stars, paths, cycles and trees. Let \( M_t \) denote a matching of size \( t \). Then the well-known Erdős-Ko-Rado Theorem [6] can be stated as follows.

**Theorem 2.1** (Erdős-Ko-Rado) Let \( n, r \geq 2 \). If \( n < 2r \), then \( ex_r(n, M_2^+) = \binom{n}{r} \). While if \( n \geq 2r \), then \( ex_r(n, M_2^+) = \binom{n-1}{r-1} \). For \( n > 2r \), equality holds above only for a star.

A recent result by Frankl [7] can be stated as

**Theorem 2.2** Let \( r, t \geq 1 \) and \( n \geq (2t + 1)r - t \). Then \( ex_r(n, M_{t+1}^+) = \binom{n}{r} - \binom{n-t}{r} \). Equality holds only for families isomorphic to \( \{ e \in \binom{[n]}{r} : e \cap [t] \neq \emptyset \} \).

Let \( S_t = K_{1,t} \). Then \( ex_3(n, S_2^+) \) and \( ex_4(n, S_2^+) \) are determined by Erdős and Sós [18]. Frankl and Chung [4] gave a result on \( ex_3(n, S_t^+) \). And there are results on \( ex_r(n, P_t^+) \) (\( r \geq 3, t \geq 4 \)) and \( ex_3(n, C_t^+) \) for sufficiently large \( n \) in [9–11].

A set of vertices in a hypergraph \( F \) containing exactly one vertex from every edge of \( F \) is called a crosscut. Let \( \sigma(F) \) be the minimum size of a crosscut of \( F \) if it exists, i.e.,

\[
\sigma(F) := \min \{|X| : X \subset V(F), \forall e \in F, |e \cap X| = 1\}
\]

if such an \( X \) exists.

**Theorem 2.3** [8, 13] Fix \( r \geq 3 \) and a forest \( G \). Then \( ex_r(n, G^+) \sim (\sigma(G^+) - 1) \binom{n}{r-1} \).

For complete bipartite graph, there are some results for \( ex_3(n, K_{3,t}^+) \) in [13].
3 Plan

1. This semester I plan to read the following articles:

- A survey of Turán problems for expansions [16], by Dhruv Mubayi and Jacques Verstraëte.
- Turán problems and shadows II: trees [12], by A. Kostochka and D. Mubayi.
- Turán problems and shadows III: bipartite graphs [13], by A. Kostochka and D. Mubayi.

2. I will consider some open problems in the survey paper [16] listed below, and decide one to work on in the next semester.

**Problem 1** If $G$ has treewidth 2, determine whether $ex_3(n, G^+) \sim (\sigma(G) - 1)\binom{n}{2}$.

**Problem 2** Determine the order of magnitude of $ex_3(n, G^+)$ when $G$ is a graph with $\sigma(G^+) = 3$.

**Problem 3** Determine whether $ex_3(n, G^+) = O(n^2)$ for every graph $G$ with an acyclic 3-coloring.

**Problem 4** If $G$ is a 2-degenerate graph, is $ex_3(n, G^+) = O(n^2)$?

**Problem 5** Determine $\lim_{n \to \infty} \frac{ex_r(n, T^+)}{\binom{n}{r-1}}$ when $T$ is the shadow of an $r$-uniform tight tree.

References


