An Analysis of Phase Noise and Fokker-Planck Equations *

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Abstract

A local moving orthonormal transformation has been introduced to rigorously study phase noise in stochastic differential equations (SDE's) arising from nonlinear oscillators. A general theory of phase and amplitude noise equations and its corresponding Fokker-Planck equations are derived to characterize the dynamics of phase and amplitude error. As an example, a van der Pol oscillator is considered by using the general theory.

1 Introduction

Phase noise in nonlinear oscillators is very important in circuit design and other areas such as optics. For example, it is known that timing jitter in circuit design is caused by phase noise [9] [15]. Mathematically, nonlinear oscillators can often be described by nonlinear autonomous differential equations with periodic orbits (limit cycles in the plane) that are orbitally asymptotically stable. We note that any solutions near an orbitally asymptotically stable periodic orbit in phase space will stay close to the periodic orbit and approach the periodic orbit in phase space with asymptotic phase [7]. However, noise is unavoidable in practice and is often modeled by additional stochastic terms in the nonlinear differential equations. In Figure 1, we have an asymptotically stable periodic orbit Γ (solid line) in phase space with least period $T > 0$ of an unperturbed nonlinear oscillator. The orbit returns to its initial state , after time T. However, a perturbed solution does not return to the starting point after the same time T due to random perturbations. Thus, natural rhythm of the oscillator is disturbed. Phase noise refers to the variations in the oscillation frequency, and jitter is the fluctuations in the period.

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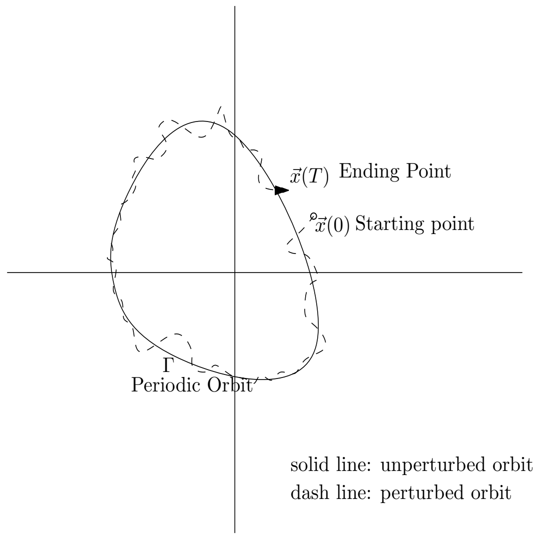

Figure 1: Perturbations near a orbitally asymptotically stable limit cycle Γ. Solutions will not return to their starting states after period T

There is a large literature dealing with phase noise problems (see, for example, [16], [17], [13], [10], [4] and references therein). However, it is indicated in [4], that theoretical understanding in the subject is rather incomplete. The main difficulty is how to completely separate phase and amplitude components in the error analysis in the nonlinear dynamics under random perturbations, which is the goal of this paper.

Standard approaches to study phase noise are largely based on linearizations of the nonlinear dynamic systems. The main idea is to use linear parts in Taylor expansions to replace the nonlinear terms near the unperturbed orbits. The key assumption for this idea to be useful is that the difference between perturbed and unperturbed solutions remains small. However it has been discussed in both [4] and [11] that the deviation of the perturbed solution from the unperturbed solution can grow to infinitely large even for orbits that are orbitally exponentially asmptotically stable. This is the reason that why linearization strategies can lead to incorrect characterization of the real phenomena in phase noise analysis.

Recently, two different nonlinear approaches have been proposed. One is based on Floquet theory and by considering a delay phase coordinate to characterize the leading contributions of the phase noise [4]. The delay phase coordinate satisfies a stochastic differential equation depending on the largest eigenvalue (must be 1 to sustain the periodic orbit) of the transition (monodromy) matrix of the linearized system and its corresponding eigenfunction. Phase noise from other components of spectrum of the transition matrix decays to zero eventually if one assumes that the random perturbations exist for only a finite time of period.

The second approach is based on the Fokker-Planck equation associated with the SDE. The standard SDE theory suggests that every diffusive SDE including the SDE governing the oscillator considered in this paper corresponds to a parabolic equation (Fokker-Planck equation, also called Kolmogorov equation in many literature) which is used to describe the evolution of the proability density function of the stochastic processes. One would then directly estimate the probability density function of the phase noise by study its Fokker-Planck equation. In [11], asymptotic analysis is carried out based on scale separation assumptions in the model separating the leading component from the Fokker-Planck equation. In addition, one assumes that trajectories are attracted to the limit cycle more than they are diffused by the noise. Under this assumption, one obtains a separation of the phase noise equations from the amplitude error component. Then the resulting simplified Fokker-Planck equation can be solved analytically by standard PDE methods. However, both approaches do not provide a complete and rigorous separation of the phase and amplitude noise.

In this paper, we present a different approach. By using a local moving transformation based on the periodic orbit (vector bundle structure over the periodic orbit) to develop a general theory that completely separates the phase and amplitude noise. The transformation enables us to rigorously derive dynamic equations explicitly for the phase noise and amplitude error. Both phase and amplitude noise remain as diffusion processes as one expects. The associated Fokker-Planck equation follows from the standard SDE theory to characterize the evolution of the probability density. We further apply the general theory to a van der Pol oscillator, a prototype of practical oscillators. And the results can be used to explain many interesting phenomena observed in practice.

The arrangement of the paper is as follows. In Section 2, we introduce the moving orthonormal coordinate system to explicitly separate the phase and amplitude representations. We state and prove the main results of this paper in Section 3. An example of analyzing van der Pol oscillator by the general theory is shown in Section 4. For reader' convenience, specially those who don't have strong background in SDE's, we insert some basic knowledge on the subject at where it will be used throughout the paper.

2 Moving orthonormal coordinate systems

In this section, we review a local moving orthonormal coordinate system along a periodic orbit of a dynamical system in an $2 \leq n < \infty$ dimensional Euclidean space [7]. For simplicity, we mainly consider $n = 2$ in this paper even thought it is valid for any n. Although the analysis for higher dimensional oscillators may be different, and more difficult in many cases, the general theory developed in Section 3 can be extended to the general case. We will state such results at the end of the section.

We start with the following autonomous system in the plane,

$$\dot{u}(t) = f(u(t)), \tag{1}$$

where $f : \mathbf{R}^2 \to \mathbf{R}^2$ is C^r, $r \geq 2$. Assume that it has a periodic orbit

$$\Gamma = \{u(t) \in \mathbf{R}^2, 0 \leq y \leq T\}, \tag{2}$$

where $T > 0$ is the least period of $u(\cdot)$. We are interested in the case that Γ is orbitally stable, which means for any given $\epsilon > 0$, there exits a $\delta(\epsilon) > 0$ such that if the distance between the starting state $u(0)$ and Γ is smaller than $\delta(\epsilon)$, then the distance between $u(t)$ and Γ is less than ϵ for all $t > 0$. More precisely,

$$\mathrm{dist}(u(t), \Gamma) \leq \epsilon, \quad t \geq 0,$$

if

$$\mathrm{dist}(u(0), \Gamma) \leq \delta(\epsilon).$$

We now consider a perturbed system of (1)

$$\dot{x}(t) = f(x(t)) + g(x(t), t), \tag{3}$$

where $g(x, t)$ is a small time dependent deterministic perturbation. In this paper, we use $x(t)$ to denote solutions of the perturbed system and $u(t)$ for the unperturbed system.

Since Γ is orbitally stable, solutions of (3) near Γ stay close to Γ. Consequently, one can introduce a local moving orthonormal coordinate system along Γ in the following manner. Note that Γ is C^r diffeomorphic to the unit circle $\mathbf{S^1}$ and the coordinate system we will introduce is a vector bundle structure over $\mathbf{S^1}$. At each point on the periodic orbit, the normalized tangent direction is

$$v(t) = \frac{1}{r} \begin{bmatrix} f_1(u(t)) \\ f_2(u(t)) \end{bmatrix}, \tag{4}$$

where $r = \sqrt{f_1^2 + f_2^2}$. The corresponding outward normal direction is

$$z(t) = \frac{1}{r} \begin{bmatrix} f_2(u(t)) \\ -f_1(u(t)) \end{bmatrix}. \tag{5}$$

Using this moving orthonormal coordinate system, as shown in Figure 2, any point x near Γ can be transformed into a new representation by using the following transformation ψ,

$$x = \psi\left(\begin{bmatrix} \theta \\ \rho \end{bmatrix}\right) = u(\theta) + z(\theta)\rho, \tag{6}$$

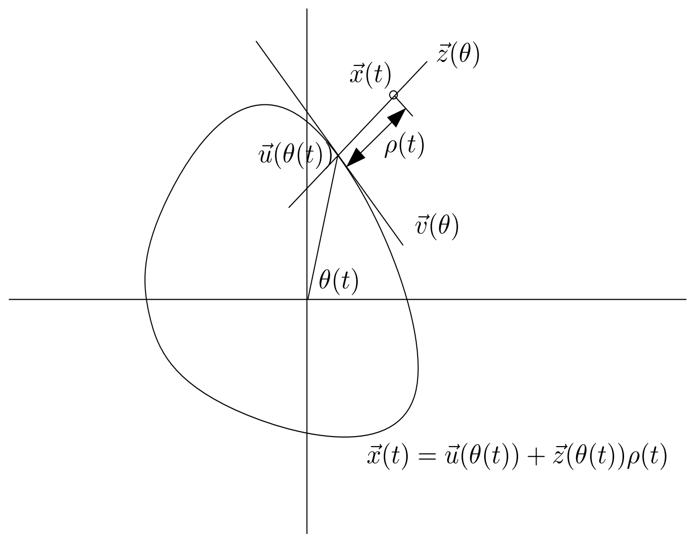

Figure 2: Transform ψ

where $\theta = t (\mathrm{mod}\, T) \in \mathbf{S^1}$, $u(\theta) = u(t)$ is the unique point on the periodic orbit Γ such that x lies in the normal space at $u(\theta)$, and ρ is the signed distance between x and $u(\theta)$. Note that, if $x(t)$ is a solution of the perturbed equation 3, then in terms of the new coordinates we have the following:

$$x(t) = \psi\left(\begin{bmatrix} \theta(t) \\ \rho(t) \end{bmatrix}\right) = u(\theta(t)) + z(\theta(t))\rho(t). \tag{7}$$

In practice, $\theta(t) - t$ corresponds to the phase error and $\rho(t)$ is the amplitude error. Obviously, the diffeomorphism ψ transforms a perturbed solution $x(t)$ into $[\theta(t), \rho(t)]$ which provides the phase of $x(t)$ and its associated amplitude error from Γ. Furthermore, this would allow us to explicitly study the phase and amplitude errors of (3) from (1).

We would like to point out that the above representations are different from the traditional understandings of local orthogonal projections, which normally result in two orthogonal components. Under transformation (7), $u(\theta(t))$ is always on the periodic orbit Γ and is not orthogonal to $z(\theta(t))$. However, $z(\theta(t))$ is orthogonal to to the tangent vector at $u(\theta(t)) \in \Gamma$.

If one assumes that f and g are C^r, $r \geq 2$. Then there exists a $\delta > 0$, such that the transformation ψ defined by (6) is a C^r diffeomorphism from $\mathbf{S^1} \times [-\delta, \delta]$ onto its image. Furthermore, the perturbed equation (3) can be expressed in the new coordinate system (θ, ρ):

$$\begin{cases} \dot{\theta} = \frac{p}{r}(f_1(\bar{f}_1 + \bar{g}_1) + f_2(\bar{f}_2 + \bar{g}_2)), \\ \dot{\rho} = \frac{1}{r}(-f_1(\bar{f}_2 + \bar{g}_2) + f_2(\bar{f}_1 + \bar{g}_1)). \end{cases} \tag{8}$$

where

$$f = f(u(\theta(t))),$$

$$\bar{f} = f(x(t)) = f(u(\theta(t)) + z(\theta(t))\rho(t)),$$
$$\bar{g} = g(x(t), t) = g(u(\theta(t)) + z(\theta(t))\rho(t), t).$$

$$w = \frac{f_1 f_2' - f_2 f_1'}{r^2}, \quad p = (r + w\rho)^{-1},$$

The proof of the above results can be found in [7] except for the explicit formulae in (8) which can be obtained from direct substitution. We also refer the reader to [3] for a similar transformation in infinite dimensional space.

Before we proceed to the main results of the paper, we state some useful relationships that are well known results in differential geometry and can also be easily verified.

$$\frac{dv(\theta)}{dt} = -wz(\theta), \qquad \frac{dz(\theta)}{dt} = wv(\theta). \tag{9}$$

3 Moving orthonormal coordinate systems under noise

As discussed in the introduction, noise is often unavoidable and un-predictable in practice. To model the influence of this perturbation, random variables are introduced in the system.

$$\frac{dX(t)}{dt} = f(X) + g(X, t) + a(X)\zeta_t, \tag{10}$$

where ζ_t is a time dependent random variable, and a a given 2×2 diagonal matrix function. As a convention in the paper, we use capital letters to represent stochastic variables.

Furthermore, if ζ_t is normally (Gaussian) distributed, equation (10) is usually written in the following standard SDE format,

$$dX(t) = f(X)dt + g(X, t)dt + a(X)dW_t, \tag{11}$$

where $W_t = [W_t^1, W_t^2]' \in \mathbf{R}^2$ is a 2-dimensional independent Brownian motion, and dW_t is its increament to model the Gaussian random perturbation $\zeta_t dt$ which is called white noise. The term $a(X)dW_t$ is usually called diffusion, and $(f(X) + g(X, t))dt$ the drift term.

It is well known that Brownian motions are continuous but not differentiable. Hence the SDE's (11) can not be understood as a system of traditional ODE's. Instead, they are defined in the Ito sense, which means that $X(t)$ is a random process satisfying the following integral equation,

$$X(t) = X(0) + \int_0^t (f(X(s)) + g(X(s), s))ds + \int_0^t a(X(s))dW_s.$$

The last term is an Ito integral, which is defined as

$$\int_0^t a(X(s))dW_s = st - \lim_{n\to\infty} \sum_{i=1}^n a(X(s_{i-1}))(W_{s_i} - W_{s_{i-1}}),$$

where $st - \lim$ means convergence in the probability sense. Such defined $X(t)$ is called Ito process in the stochastic literature.

One of the most significiant difference between Ito SDE's and the standard differential equations is that Ito SDE's have a different chain rule in its calculus, which is best described by the following Ito formula [1].

Ito Formula: Let $v(y,t)$ denote a continuous function defined on $\mathbf{R}^n \times [t_0, T]$ with values in $\mathbf{R}^m$ and with the continuous partial derivatives v_t, v_{y_i} and $v_{y_i y_j}$. If the $n-$dimensional stochastic process $Y(t)$ is defined on $[t_0, T]$ by

$$dY(t) = l(Y,t)dt + k(Y,t)dW_t, \tag{12}$$

then $Z(t) = v(Y(t), t)$ defined on $[t_0, T]$ with a given initial condition $Z(t_0) = v(X(t_0), t_0)$ is also a Ito stochastic process satisfying a stochastic differential equation,

$$\begin{aligned} dZ(t) &= (v_t(Y,t) + v_y(Y,t)l(Y,t) + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n v_{y_i y_j}(t,Y)(kk')_{ij})dt \qquad (13) \\ &\quad + v_y(Y,t)k(Y,t)dW_t, \end{aligned}$$

where k' is the transpose of k.

We notice that the term $\frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n v_{y_i y_j}(t,Y)(kk')_{ij}$ is new comparing to the standard chain rule, and it involves the second order derivatives of v and the diffusion coefficient k. This is mainly due to the following basic facts of Brownian motions,

$$E(dW_t^i dW_t^i) = dt, \quad E(dW_t^i dW_t^j) = 0,$$

where $E(\cdot)$ denotes the expecion of a random variable. The second identity describes that different Brownian motions have independent increaments. The first one states that the increaments of a Browian motion (Gaussian random variable) have variance dt, which implies that the product of the diffusion term in (12) generate a term containing dt.

With such understandings of noisy system (11), we study its phase and amplitude noise. Our strategy is to apply the transform (6) and follow the deterministic perturbation theory [7]. In order to use the transform (6), we assume that the Ito process $X(t)$ stays close to the periodic orbit Γ almost surely (or with large probability). More precisely, we assume that with probability 1 (or $1 - \beta$, where β is a small positive number),

$$\text{dist}(X(t), \Gamma) \le \gamma, \quad 0 \le t \le T,$$

for some small positive number γ. Time T may be finite or infinite.

Using the transformation ψ, which is smooth and deterministic, defined in the previous section, we can represent $X(t)$ by

$$X(t) = \psi\left(\left[\begin{array}{c}\Theta(t)\\ \Lambda(t)\end{array}\right]\right) = u(\Theta(t)) + z(\Theta(t))\Lambda(t), \tag{14}$$

where $\Theta(t)$ and $\Lambda(t)$ are random functions describing the phase noise, $\Theta(t) - t$, and its associated amplitude noise of (11). They satisfy the following dynamic equations.

Theorem 1 *Assume that the solution $X(t)$ of*

$$dX(t) = f(X)dt + g(X,t)dt + a(X)dW_t, \tag{15}$$

almost surely stays close enough to the periodic orbit Γ, and both f and g are C^r smooth functions, $r \geq 2$. Then under the transform ψ, $[\Theta(t), \Lambda(t)]$ remain as Ito processes and satisfy the following Ito stochastic differential equations,

$$d\left[\begin{array}{c}\Theta(t)\\ \Lambda(t)\end{array}\right] = h(\Theta,\Lambda)dt + c(\Theta,\Lambda)dW_t, \tag{16}$$

where the coefficients $h \in \mathbf{R}^2$ and $c \in \mathbf{R}^{2\times 2}$ are defined by

$$\left\{\begin{array}{lll} h_1 & = & \frac{p}{r}(f_1(\bar{f}_1 + \bar{g}_1) + f_2(\bar{f}_2 + \bar{g}_2) + \frac{1}{2r}\frac{\partial p}{\partial \theta}((f_1\bar{a}_{11})^2 + (f_2\bar{a}_{22})^2) \\ & & +\frac{1}{2r}wpf_1f_2(\bar{a}_{22}^2 - \bar{a}_{11}^2)), \\ h_2 & = & \frac{1}{r}(-f_1(\bar{f}_2 + \bar{g}_2) + f_2(\bar{f}_1 + \bar{g}_1) + \frac{1}{2r}wp((f_1\bar{a}_{11})^2 + (f_2\bar{a}_{22})^2)), \end{array}\right. \tag{17}$$

and

$$c = \left[\begin{array}{cc} \frac{p}{r}f_1\bar{a}_{11} & \frac{p}{r}f_2\bar{a}_{22}, \\ \frac{1}{r}f_2\bar{a}_{11} & -\frac{1}{r}f_1\bar{a}_{22} \end{array}\right], \tag{18}$$

where $\bar{a} = a(u(\Theta) + z(\Theta)\Lambda))$.

Proof of Theorem 1: We first show that $[\Theta(t), \Lambda(t)]$ remain as Ito processes. By assumption that the solution $X(t)$ stays closely to the periodic orbit, which implies that the transformation ψ, which is is a C^r diffeomorphism with $r \geq 2$, is valid. By Ito's formula (13), this implies that the stochastic processes

$$\left[\begin{array}{c}\Theta(t)\\ \Lambda(t)\end{array}\right] = \psi^{-1}(X(t)) = \left[\begin{array}{c}\theta(X(t))\\ \rho(X(t))\end{array}\right] \tag{19}$$

are also Ito processes and satisfy the following equations

$$\left\{\begin{array}{lll} d\Theta & = & \theta_{x_1}dX_1 + \theta_{x_2}dX_2 \\ & & +\frac{1}{2}(\theta_{x_1x_1}(dX_1)^2 + \theta_{x_1x_2}dX_1dX_2 + \theta_{x_2x_2}(dX_2)^2), \\ d\Lambda & = & \rho_{x_1}dX_1 + \rho_{x_2}dX_2 \\ & & +\frac{1}{2}(\rho_{x_1x_1}(dX_1)^2 + \rho_{x_1x_2}dX_1dX_2 + \rho_{x_2x_2}(dX_2)^2). \end{array}\right. \tag{20}$$

By Ito's formula (13), one obtains that

$$(dX_1)^2 = \bar{a}_{11}^2 dt, \quad dX_1 dX_2 = 0, \quad (dX_2)^2 = \bar{a}_{22}^2 dt.$$

We substitute them into (20) to obtain

$$\left\{ \begin{array}{lcl} d\Theta & = & \left[\nabla\theta \cdot (\bar{f} + \bar{g}) + \frac{1}{2}(\theta_{x_1x_1}\bar{a}_{11}^2 + \theta_{x_2x_2}\bar{a}_{22}^2)\right]dt + (\nabla\theta)'\bar{a}dW_t \\ d\Lambda & = & \left[\nabla\rho \cdot (\bar{f} + \bar{g}) + \frac{1}{2}(\rho_{x_1x_1}\bar{a}_{11}^2 + \rho_{x_2x_2}\bar{a}_{22}^2)\right]dt + (\nabla\rho)'\bar{a}dW_t \end{array} \right. \tag{21}$$

Therefore, we can write this as (16) with h, c are coefficients to be determined.

Because of (14) and Ito's formula, we also have

$$dX(t) = \psi_\theta d\Theta + \psi_\rho d\Lambda + \frac{1}{2}\left[\psi_{\theta\theta}(d\Theta)^2 + \psi_{\theta\rho} d\Theta d\Lambda + \psi_{\rho\rho}(d\Lambda)^2\right]. \tag{22}$$

Using the facts that

$$\psi_{\rho\rho} = 0, \quad \psi_{\theta\rho} = z_\theta$$

and Ito's formula again, we have

$$d\Theta d\Lambda = (c_{11}c_{21} + c_{12}c_{22})dt, \quad d\Theta d\Theta = (c_{11}^2 + c_{12}^2)dt,$$

we then obtain

$$\begin{array}{rcl} dX(t) & = & \psi_\theta d\Theta + \psi_\rho d\Lambda + \frac{1}{2}\left[\psi_{\theta\theta}(c_{11}^2 + c_{12}^2) + z_\theta(c_{11}c_{21} + c_{12}c_{22})\right] dt \\ & = & \left[h_1\psi_\theta + h_2 z + \frac{1}{2}\psi_{\theta\theta}(c_{11}^2 + c_{12}^2) + \frac{1}{2}z_\theta(c_{11}c_{21} + c_{12}c_{22})\right] dt \\ & & + \left[\psi_\theta c_{11} + z c_{21}\right] dW_t^1 + \left[\psi_\theta c_{12} + z c_{22}\right] dW_t^2. \end{array} \tag{23}$$

By matching the coefficients of (11) and (23), we have the following system for the coefficients h and c,

$$\left\{ \begin{array}{l} \bar{f} + \bar{g} = h_1\psi_\theta + h_2 z + \frac{1}{2}\psi_{\theta\theta}(c_{11}^2 + c_{12}^2) + \frac{1}{2}z_\theta(c_{11}c_{21} + c_{12}c_{22}), \\ \bar{a} = \left[\ \psi_\theta c_{11} + z c_{21}, \quad \psi_\theta c_{12} + z c_{22}\ \right]. \end{array} \right. \tag{24}$$

From the definition of the diffeomorphism ψ (6), it is easy to verify that

$$\psi_\theta = \frac{v}{p},$$

and

$$\psi_{\theta\theta} = -\frac{1}{p^2}p_\theta v - \frac{1}{p}wz.$$

Then solving the coefficient equations (24), we obtain (17) and (18), which completes the proof.

Remarks:

1. We note that equation (16) is reduced to (8) if the stochastic perturbations vanish.

2. If the periodic solution Γ is orbital stable and the perturbations are small, the solution $X(t)$ usually stays close to Γ with large probability in a relative large time scale. We will demonstrate this in the example shown in Section 4.

As we have already seen that under the new moving coordinate systems, the phase and amplitude error are Ito stochastic processes satisfying SDE's (16). This implies that for every different realization of the Brownian motion path, there is a different dynamic process describing the phase and amplitude evolutions. Therefore, for such stochastic processes, it is often more desirable to understand their statistical properties, such as the probability distribution function, instead of each individual realization. There is a well developed diffusion theory (see, for example, [6] [14]) for these issues. The probability density function of a stochastic processes satisfies a parabolic equation, called Fokker-Planck equation or forward Kolmogorov equation, which is stated next.

Let $p(y,t)$ be the probability density function of the random process $Y(t)$ defined by (12), i.e.

$$p(y,t) = \text{Prob}\{Y(t) = y\}.$$

Then $p(y,t)$ satisfies the following evalution equation

$$p_t = -(lp)_y + \frac{1}{2}(kk'p)_{yy}.$$

Following this result, if one denotes $p(\theta,\lambda,t)$ as the probability density function of $[\Theta(t),\Lambda(t)]$, i.e.

$$p(\theta,\lambda,t) = \text{Prob}\{(\Theta(t),\Lambda(t)) = (\theta,\lambda)\},$$

then the associated Fokker-Planck equation can be directly obtained. And we state it in the next theorem.

Theorem 2 *The probability density $p(\theta,\lambda,t)$ for the processes $[\Theta(t),\Lambda(t)]$ satisfies the following evolution equation,*

$$\begin{aligned} p_t &= -\nabla\cdot(hp) \\ &\quad +\tfrac{1}{2}(((c_{11}^2+c_{12}^2)p)_{\theta\theta} + 2((c_{11}c_{21}+c_{12}c_{22})p)_{\theta\lambda} + ((c_{21}^2+c_{22}^2)p)_{\lambda\lambda}. \end{aligned} \tag{25}$$

And if the starting point of $X(t)$ is at $u(\theta_0)+z(\theta_0)\lambda_0$, the initial condition for (25) is

$$p(\theta,\lambda,0) = \delta(\theta-\theta_0)\delta(\lambda-\lambda_0), \tag{26}$$

where δ is the standard Dirac function.

We close this section by noting that the results discussed here can be generalized to the general n dimensional systems. We state them in the Appendix.

4 An example of van der Pol oscillator

In this section, we use the general theory developed in the previous section to analyze a model problem, which is the same problem considered in [11], a nonlinear circuit of van der Pol type of oscillator. We refer the reader to [11] for the actual circuit design and how noise should be modeled in the system.

The van der Pol oscillator satisfies the following second order differential equation,

$$\ddot{q} - \alpha(1 - \dot{q}^2)\dot{q} + q = 0, \tag{27}$$

By introducing a new variable $u = [q, \dot{q}]'$, the above equation (27) is converted into the following first order system,

$$\begin{cases} \dot{u}_1 = u_2, \\ \dot{u}_2 = -u_1 + \alpha(1 - u_2^2)u_2. \end{cases} \tag{28}$$

In applications, $\alpha > 0$ is a small parameter. It is known that this system has a vertical Hopf bifurcation from the origin at the parameter $\alpha = 0$, and for every small $\alpha > 0$ there exits a unique orbitally exponentially stable limit cycle denoted by Γ_α. Note that this periodic orbit is not close to the origin [2]. We shall construct a moving local coordinate system along Γ_α as described in Section 2.

To introduce noise into the system, we consider the following noisy van der Pol oscillator equation:

$$\begin{cases} \dot{X}_1 = X_2, \\ \dot{X}_2 = -X_1 + \alpha(1 - X_2^2)X + \epsilon dW_t. \end{cases} \tag{29}$$

where dW_t is a 1−dimensional white noise, and ϵ is a small positive number. In application, the magnitude of ϵ is of the same order as α. However, in order to better illustrate our analysis, we distinguish them in the following derivation. We will analyze the phase noise with $\epsilon \sim \alpha$ at the end of this section. We assume that both α and ϵ are small enough in this paper to ensure the asymptotic analysis to be carried out.

4.1 Approximation to the periodic orbit

In order to use the local moving coordinate system and the transformation ψ, we need to understand the periodic solution Γ_α. However, we are not able to get an explicit (analytic) formula for the periodic orbit Γ_α. Hence, we will use asymptotic analysis and the method of averaging to study the leading terms of Γ_α which is based on the method as described in [2].

We first transform u into the polar coordinate system $[\eta, \omega]$. where

$$u_1 = \eta\cos\omega, \quad u_2 = \eta\sin\omega.$$

Then system (28) is transformed into

$$\begin{cases} \dot{\eta} = \alpha\eta(1-\eta^2\sin^2\omega)\sin^2\omega, \\ \dot{\omega} = -1+\alpha(1-\eta^2\sin^2\omega)\cos\omega\sin\omega. \end{cases} \tag{30}$$

Assume that $0 < \alpha \ll 1$, we define new variables

$$\bar{\eta} = \eta + \alpha v_1(\eta,\omega), \quad \bar{\omega} = \omega + \alpha v_2(\eta,\omega), \tag{31}$$

where v_1, v_2 are unknown functions to be determined. Equivalently, the inverse transform of (31) is

$$\eta = \bar{\eta} + \alpha\bar{v}_1(\bar{\eta},\bar{\omega}), \quad \omega = \bar{\omega} + \alpha\bar{v}_2(\bar{\eta},\bar{\omega}), \tag{32}$$

From (31) and equation (30), we obtain that

$$\begin{aligned} \dot{\bar{\eta}} &= \dot{\eta} + \alpha\left(\tfrac{\partial v_1}{\partial \eta}\dot{\eta} + \tfrac{\partial v_1}{\partial \omega}\dot{\omega}\right) \\ &= \alpha\left(\eta(1-\eta^2\sin^2\omega)\sin^2\omega - \tfrac{\partial v_1}{\partial \omega}\right) + O(\alpha^2), \end{aligned} \tag{33}$$

and

$$\begin{aligned} \dot{\bar{\omega}} &= \dot{\omega} + \alpha\left(\tfrac{\partial v_2}{\partial \eta}\dot{\eta} + \tfrac{\partial v_2}{\partial \omega}\dot{\omega}\right) \\ &= -1 + \alpha\left((1-\eta^2\sin^2\omega)\sin\omega\cos\omega - \tfrac{\partial v_2}{\partial \omega}\right) + O(\alpha^2). \end{aligned} \tag{34}$$

As in the method of averaging, one defines a function v_1 by

$$\frac{\partial v_1}{\partial \omega} = \eta(1-\eta^2\sin^2\omega)\sin^2\omega + C,$$

where

$$C = \frac{1}{2\pi}\int_0^{2\pi} \eta(1-\eta^2\sin^2\omega)\sin^2\omega d\omega = \frac{1}{2}\eta - \frac{3}{8}\eta^3.$$

Similarly, we define a function v_2 by

$$\frac{\partial v_2}{\partial \omega} = (1-\eta^2\sin^2\omega)\sin\omega\cos\omega + D,$$

where

$$D = \frac{1}{2\pi}\int_0^{2\pi} (1-\eta^2\sin^2\omega)\sin\omega\cos\omega d\omega = 0.$$

Substituting the functions v_1, v_2 into (33) and (34), we have

$$\begin{cases} \dot{\bar{\eta}} = \alpha(\frac{1}{2}\eta - \frac{3}{8}\eta^3) + O(\alpha^2) \\ \dot{\bar{\omega}} = -1 + O(\alpha^2). \end{cases} \tag{35}$$

By (32), we obtain

$$\begin{cases} \dot{\bar{\eta}} = \alpha(\frac{1}{2}\bar{\eta} - \frac{3}{8}\bar{\eta}^3) + O(\alpha^2) \\ \dot{\bar{\omega}} = -1 + O(\alpha^2). \end{cases} \tag{36}$$

Since α is small, we have the following approximation:

$$[\xi, \phi] = [\bar{\eta}, \bar{\omega}] + O(\alpha)$$

and

$$\begin{cases} \dot{\xi} = \alpha(\frac{1}{2}\xi - \frac{3}{8}\xi^3) \\ \dot{\phi} = -1, \end{cases} \tag{37}$$

For the equilibria or periodic orbits, we consider:

$$\frac{1}{2}\xi - \frac{3}{8}\xi^3 = 0.$$

This implies

$$\xi = 0, \quad \text{or} \quad \xi = \frac{2}{\sqrt{3}}.$$

Obviously, $\xi = \frac{2}{\sqrt{3}}$ and $\phi = -\theta$ are the leading term approximations to the periodic orbit Γ_α. This implies that on Γ_α,

$$\begin{cases} \eta(\theta) = \frac{2}{\sqrt{3}} + O(\alpha) \\ \omega(\theta) = -\theta + O(\alpha). \end{cases} \tag{38}$$

Plugging the above approximations into the polar coordinate formula, we have

$$\begin{cases} u_1(\theta) = \eta \cos\omega = \frac{2}{\sqrt{3}} \cos\theta + O(\alpha) \\ u_2(\theta) = \eta \sin\omega = -\frac{2}{\sqrt{3}} \sin\theta + O(\alpha). \end{cases} \tag{39}$$

4.2 Analysis of stochastic van der Pol oscillator

In this section, we apply the general theory developed in Section 3 to (29) in the following setting:

$$f(u) = \begin{bmatrix} u_2 \\ -u_1 + \alpha(1 - u_2^2)u_2^2 \end{bmatrix}, \quad g = 0, \quad a = \begin{bmatrix} 0 & 0 \\ 0 & \epsilon \end{bmatrix}.$$

We use the transform (6) to get

$$X(t) = \psi(\Theta, \Lambda),$$

which gives the drift term of the perturbed system:

$$\bar{f} = \bar{f}(X(t)) = \bar{f}(\Theta, \Lambda) = \begin{bmatrix} (1 - \frac{\Lambda}{r})u_2(\Theta) \\ (-(1 - \frac{\Lambda}{r})u_1(\Theta) + O(\alpha)). \end{bmatrix}.$$

By (17) and (18), we have

$$\begin{cases} h_1 = \frac{p}{r}(f'\bar{f} + \frac{1}{2r}p_\theta(\epsilon f_2)^2 + \frac{1}{2r}wpf_1f_2\epsilon^2), \\ h_2 = \frac{1}{r}(-f_1\bar{f}_2 + f_2\bar{f}_1 + \frac{1}{2r}wp(\epsilon f_2)^2), \end{cases} \tag{40}$$

and

$$\begin{cases} c_{11} = 0, \\ c_{12} = \epsilon(-u_1 + \alpha(1-u_2^2)u_2)\frac{p}{r}, \\ c_{21} = 0, \\ c_{22} = -\epsilon\frac{u_2}{r}. \end{cases} \tag{41}$$

Using the asymptotic expansion (39), one obtains

$$\begin{array}{l} r = \frac{2}{\sqrt{3}} + O(\alpha), \quad w = -1 + O(\alpha), \\ p(\theta, \rho) = (r + w\rho)^{-1} = \frac{\sqrt{3}}{2}(1 + \frac{\sqrt{3}}{2}\rho + O(\alpha + \rho^2), \quad p_\theta = O(\alpha). \end{array}$$

Therefore, applying Theorems 1 and 2, we achieve the following results.

Theorem 3 *The phase Θ and amplitude noise Λ along the periodic orbit Γ_α defined by*

$$X(t) = \psi\left(\left[\begin{array}{c} \Theta(t) \\ \Lambda(t) \end{array}\right]\right), \tag{42}$$

where $X(t)$ satisfy

$$\begin{cases} \dot{X}_1 = X_2, \\ \dot{X}_2 = -X_1 + \alpha(1 - X_2^2)X + \epsilon dW_t, \end{cases} \tag{43}$$

remain as Ito's processes and satisfy the following stochastic equations

$$\begin{cases} d\Theta &=& (1 + O(\alpha))dt + \epsilon((-\frac{2}{\sqrt{3}} - \frac{3}{4}\Lambda)\cos\Theta + O(\alpha + \Lambda^2))dW_t, \\ d\Lambda &=& (\alpha\Lambda(1 - 4\sin^2\Theta)\sin^2\Theta + O(\alpha\Lambda^2 + \alpha\epsilon^2 + \alpha^2))dt \\ && -(\epsilon\sin\Theta + O(\alpha\epsilon))dW_t, \end{cases} \tag{44}$$

The leading terms of $[\Theta, \Lambda]$, denoted by $[\tilde{\Theta}, \tilde{\Lambda}]$, satisfy

$$\begin{cases} d\tilde{\Theta} &=& dt + \epsilon((-\frac{2}{\sqrt{3}} - \frac{3}{4}\tilde{\Lambda})\cos\tilde{\Theta})dW_t, \\ d\tilde{\Lambda} &=& -\alpha\tilde{\Lambda}dt - \epsilon\sin\tilde{\Theta}dW_t. \end{cases} \tag{45}$$

Furthermore, let $p(\theta, \lambda, t)$ be the probability density function of $[\tilde{\Theta}, \tilde{\Lambda}]$. Then p satisfies

$$\begin{array}{rcl} \frac{\partial p}{\partial t} &=& -\frac{\partial}{\partial\theta}p + \frac{\partial}{\partial\lambda}(\lambda p) + \frac{1}{2}(\frac{\partial^2}{\partial\theta^2}(\epsilon^2(\frac{\sqrt{3}}{2} + \frac{3}{4}\lambda)^2\cos^2\theta)p) \\ && +2\frac{\partial^2}{\partial\theta\partial\lambda}(\epsilon^2((\frac{\sqrt{3}}{2} + \frac{3}{4}\lambda)\cos\theta\sin\theta p) + \frac{\partial^2}{\partial\lambda^2}(\epsilon^2\sin^2\theta p). \end{array} \tag{46}$$

If $X(0)$ is at $u(\theta_0) + z(\theta_0)\lambda_0$, the initial condition for (46) is

$$p(\theta, \lambda, 0) = \delta(\theta - \theta_0)\delta(\lambda - \lambda_0). \tag{47}$$

Proof: One can obtain (44) directly from Theorem 1. Here we just need to prove (45), for which we use the method of averaging for SDE's. Obviously, the leading contributions for (44) is

$$\left\{ \begin{array}{lcl} d\Theta & = & dt + \epsilon(-\frac{2}{\sqrt{3}} - \frac{3}{4}\Lambda)\cos\Theta dW_t, \\ d\Lambda & = & (\alpha\Lambda(1-4\sin^2\Theta)\sin^2\Theta dt - (\epsilon\sin\Theta)dW_t, \end{array} \right. \tag{48}$$

We define new variables $\bar{\Theta} = \Theta$ and $\bar{\Lambda} = \Lambda + \alpha v(\Theta, \Lambda)$, where $v(\theta, \lambda)$ is a deterministic function to be determined. Ito's formula gives

$$\begin{array}{rcl} d\bar{\Lambda} & = & d\Lambda + \alpha\left[v_\theta d\Theta + v_\lambda d\Lambda + O(\epsilon^2)dt\right] \\ & = & \alpha(\Lambda(1-4\sin^2\Theta)\sin^2\Theta + v_\theta + O(\alpha+\epsilon^2))dt \\ & & -(\epsilon\sin\Theta + O(\alpha\epsilon))dW_t \end{array}$$

If we define v such that

$$v_\theta = -\lambda(1-4\sin^2\theta)\sin^2\theta + E,$$

where

$$E = \frac{1}{2\pi}\int_0^{2\pi}\lambda(1-4\sin^2\theta)\sin^2\theta d\theta = -\lambda,$$

we obtain

$$\begin{array}{rcl} d\bar{\Lambda} & = & -\alpha(\Lambda + O(\alpha+\epsilon^2))dt - (\epsilon\sin\Theta + O(\alpha\epsilon))dW_t \\ & = & -\alpha(\bar{\Lambda} + O(\alpha+\epsilon^2))dt - (\epsilon\sin\tilde{\Theta} + O(\alpha\epsilon))dW_t. \end{array}$$

When α is small, the leading term satisfies of the second equation of (45). The results of (46) and (47) can be obtained directly from Theorem 2.

4.3 Discussions on the stochastic van der Pol oscillator

In this section, we discuss some interesting properties and observations associated with the van der Pol oscillator.

The following two phenomena have been observed in practice and studied in an ideal parallel LC oscillator [8] [11], which leads to the van der Pol equation.

(1) As shown in Figure 3, it is observed that when an impulse random perturbation is injected to the current in the system at the moment when the voltage crosses zero and the current reaches the peak (i.e. $\Theta = n\pi$ in phase space), the noise has maximum impact on the phase and minimum influence on the amplitude. This can be easily explained according to (45). If one takes $\Theta = n\pi$, the coefficient $(\frac{\sqrt{3}}{2} + \frac{3}{4}\Lambda)\cos\Theta$ in front of dW_t for the phase Θ achieve the maximum values in magnitude, while at the same time, the coefficient $\epsilon\sin\Theta$ in front of dW_t for the amplitude error Λ returns zero.

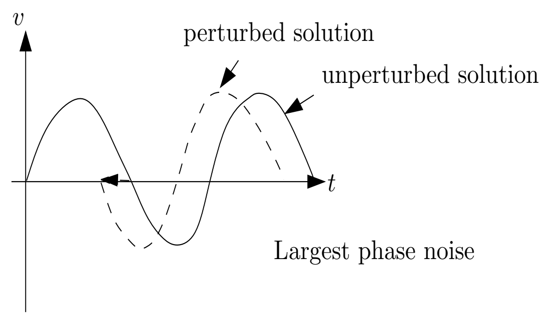

Figure 3: A impulse noise in current at the peak of current (or zero crossing of voltage).

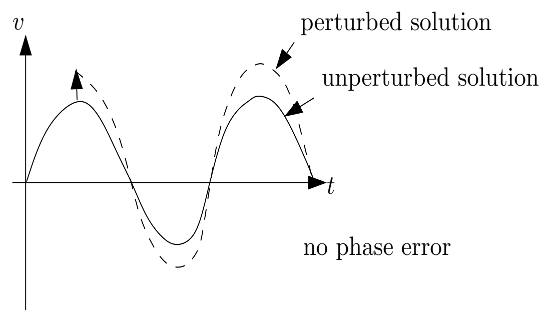

Figure 4: A impulse noise in current at the peak of voltage (or zero current).

(2) On the contrary, it is also observed that if the impulse noise is added to the current at the peak of the voltage and zero of the current (i.e. $\Theta = (n+1/2)\pi$ in the phase space), the noise has minimum impact on the phase, but maximum disturbance on the amplitude as shown in Figure 4. In this situation, the coefficient of the random perturbation for the phase Θ in (45) takes zero value and the coefficient for the amplitude error Λ gets the maximum values.

Next we examine the amplitude error and phase equations in (45) separately to reveal some of their properties. Here, we would like to point out that since the coefficients in the diffusion terms cannot be zero simultaneously, it suggests that $[\Theta, \Lambda] = [0, 0]$ is not an equilibrium of the equations, otherwise the perturbed solution will not follow the periodic orbit. Therefore, one cannot directly apply the standard stability concepts and theory developed for the zero equilibrium to this system. In fact, we cannot assume the noise type for the amplitude and phase equations directly, because they must be derived from the original noise system (29), which is regarded as a good noise model.

We start with the amplitude error. It is easy to see that the leading term in the amplitude error $\bar{\Lambda}(t)$ of (45) satisfies

$$\sup_{0 \le s \le t} |\tilde{\Lambda}(s)| \le \sup_{0 \le s \le t} |Z(s)|,$$

where Z is the well known Gaussian process defined by

$$dZ = -\alpha Z dt + \epsilon dW_t.$$

Here we note that the $\sup_{0 \le s \le t} |\tilde{\Lambda}(s)|$ refers to the largest value of $|\tilde{\Lambda}(s)|$ for all possible realizations. The standard estimates (example 6.4 in [12]) give

$$\sup_{0 \le s \le t} |Z(s)| < \frac{\epsilon^2}{\alpha} \log t,$$

which implies that

$$\sup_{0 \le s \le t} |\tilde{\Lambda}(s)| \le \frac{\epsilon^2}{\alpha} \log t,$$

if the initial amplitude error $\tilde{\Lambda}(0)$ is zero. Therefore, for any given $\beta > 0$, one obtains

$$\sup_{0 \le s \le t} |\tilde{\Lambda}(s)| < \beta$$

for all

$$t \le e^{\frac{\alpha\beta}{\epsilon^2}}.$$

We note that this estimate assures that the perturbed solutions do not leave a small neighborhood of the periodic orbit Γ for a very large time provided $\epsilon^2 = o(\alpha\beta)$, which confirms the hypothesis of the Theorem 1.

Furthermore, from equations (45), if one further approximates the leading amplitude error by

$$\tilde{\Lambda} = -\alpha \tilde{\Lambda} dt + \epsilon \sin t dW_t, \tag{49}$$

then following the standard linear SDE theory [1], which is very similar to linear ODE theory, $\tilde{\Lambda}$ is a Gaussian process with normal distribution, and the mean of $\tilde{\Lambda}(t)$ is

$$E(\tilde{\Lambda}(t)) = X(0) e^{-\alpha t},$$

and by the Ito's formula, the variance is

$$\begin{aligned} V(\tilde{\Lambda}(t)) = E((\tilde{\Lambda} - E(\tilde{\Lambda}))^2) &= \epsilon^2 \int_0^t e^{-2\alpha(t-s)} \sin^2 s ds \\ &= \frac{\epsilon^2}{2} \int_0^t e^{-2\alpha(t-s)} (1 - \cos 2s) ds. \end{aligned}$$

Using the fact that

$$\int_0^t e^{2\alpha s}\cos 2s ds = \frac{1}{2\alpha}(e^{2\alpha t}\cos 2t - 1) + \frac{1}{2\alpha^2}e^{2\alpha t}\sin 2t - \frac{1}{\alpha^2}\int_0^t e^{2\alpha s}\cos 2s ds,$$

which implies

$$\int_0^t e^{2\alpha s}\cos 2s ds = \frac{\alpha}{2(1+\alpha^2)}(e^{2\alpha t}\cos 2t - 1 + \frac{1}{\alpha}e^{2\alpha t}\sin 2t),$$

one obtains that

$$V(\tilde{\Lambda}(t)) = \epsilon^2(\frac{1}{4\alpha}(1-e^{-2\alpha t}) - \frac{\alpha}{2(1+\alpha^2)}(e^{2\alpha t}\cos 2t - 1 + \frac{1}{\alpha}e^{2\alpha t}\sin 2t)). \quad (50)$$

This suggests that

$$p(\lambda, t) = \frac{1}{\sqrt{2\pi V(\tilde{\Lambda}(t))}} e^{-\frac{\lambda^2}{2V(\tilde{\Lambda}(t))}} \quad (51)$$

is the solution of the Fokker-Planck equation associated with (49)

$$p_t = (\alpha\lambda p)_\lambda + \frac{\epsilon^2}{2}((\sin^2 t)p)_{\lambda\lambda}.$$

For small α, one has estimate

$$V(\bar{\Lambda}(t)) \leq \frac{\epsilon^2}{4\alpha}.$$

It is worth to highlight that this estimate is independent of t. Thus, for any given $\beta > 0$, the probability that $|\tilde{\Lambda}(t)| \geq \beta$ is

$$\text{Prob}(|\tilde{\Lambda}(t)| \geq \beta) \leq 2\frac{\sqrt{2\alpha}}{\epsilon\sqrt{\pi}}\int_\beta \infty e^{-\frac{2\alpha x^2}{\epsilon^2}}dx = \frac{2}{\pi}\int_{\frac{\sqrt{2\alpha}}{\epsilon}\beta}^\infty e^{-y^2}dy.$$

Particularly, if one takes $\beta = \epsilon^{1/2-\gamma}$ and $\alpha = 0.5\epsilon$, where $0 \leq \gamma < 1/2$, then

$$\text{Prob}(|\bar{\Lambda}(t)| \geq \beta) < \frac{2}{\sqrt{\pi}}\frac{\epsilon^\gamma}{e^{\epsilon^{-2\gamma}}},$$

which can be arbitrarily small provided that ϵ is small enough. This suggests that chances of the perturbed solutions leaving a small neighborhood of periodic orbit Γ in the van der Pol oscillator remain very small asymptotically provided the perturbation to the system is not too large.

Finally, we study the phase equation in (45). Following the above analysis to the amplitude error, if we assume the $\Lambda(t)$ remains in a small neighborhood of zero. we can further approximate the phase equation by

$$d\tilde{\Theta} = dt - \epsilon\frac{\sqrt{3}}{2}\cos\tilde{\Theta}dW_t, \quad (52)$$

which is a close equation for $\tilde{\Theta}$, describing the leading term behavior of the phase in the van der Pol oscillator. The probability density function $p(t,\theta)$ satisfies the associated Fokker-Planck equation,

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial \theta} p + \frac{3}{8}\epsilon^2 \frac{\partial^2}{\partial \theta^2}(\cos^2 \theta p). \tag{53}$$

By introducing new variable $\bar{\theta} = \theta - t$, and $q(\theta, t) = p(\theta + t, t)$, then q satisfies

$$\frac{\partial q}{\partial t} = \frac{3}{8}\epsilon^2 \frac{\partial^2}{\partial \theta^2}(\cos^2(\theta + t) q).$$

This clearly indicates that the phase noise is time variant which agree with many other studies including [8] and [11].

In addition, if one further simplifies the equation to

$$d\bar{\Theta} = dt - \epsilon \frac{\sqrt{3}}{2} \cos t dW_t.$$

Again, following the standard theory for linear SDE's, one can easily obtain that $\bar{\Theta} - t$ can be approximated by a Gaussian process with zero mean and variance given by

$$V(\bar{\Theta}) = \frac{3}{4}\epsilon^2 \int_0^t \cos^2 s ds,$$

which is consistent to the estimate obtained in [11].

Appendix

As mentioned before, the results described in Section 3 can be generated to the n dimensional systems. We shall state these generalizations here without giving detail derivations as they are similar to the case of $n = 2$.

We consider the following system

$$\dot{u} = f(u),$$

where both u and f are in $\mathbf{R}^n$. In deterministic situation, a system with small perturbation is

$$\dot{x} = f(x) + g(x, t),$$

where $g \in \mathbf{R}^n$. Then the solution $x(t)$ can be expressed by

$$x = \psi(\theta, \rho) = u(\theta(t)) + z(\theta(t))\rho(t), \tag{54}$$

with $z \in \mathbf{R}^{n \times (n-1)}$ and $\rho \in \mathbf{R}^{n-1}$. The columns of z form an orthonormal system of the normal space of the periodic solution u of the unperturbed system. Besides, z is also orthogonal to tangent vector f, i.e.

$$f^T z = 0.$$

The stochastically perturbed system is

$$dX(t) = (f(X) + g(X,t))dt + a(X)dW_t, \tag{55}$$

where $g \in \mathbf{R}^n$, $a(x) \in \mathbf{R}^{n\times n}$ and W_t is an n dimensional independent Brownian motion. Using the same expression (54), we have

$$X(t) = \psi(\Theta, \Lambda) = u(\Theta(t)) + z(\Theta(t))\Lambda(t), \tag{56}$$

with $\Theta \in \mathbf{R}$ and $\Lambda \in \mathbf{R}^{n-1}$.

Theorem 4 *Assume that the solution $X(t)$ of*

$$dX(t) = (f(X) + g(X,t))dt + a(X)dW_t \tag{57}$$

almost surely stays close to the periodic orbit Γ, and f is C^r with $r \geq 2$. Then transform ψ is a C^r diffeomorphism. And under the transform ψ, $[\Theta(t), \Lambda(t)]$ remain as Ito processes and satisfy the following Ito stochastic differential equations,

$$\begin{cases} d\Theta(t) = kdt + bdW_t, \\ d\Lambda(t) = hdt + cdW_t, \end{cases} \tag{58}$$

where the coefficients $k \in \mathbf{R}, b \in \mathbf{R}^n, h \in \mathbf{R}^{n-1}, c \in \mathbf{R}^{(n-1)\times n}$ satisfy the following algebraic equations

$$\begin{cases} (\bar{f} + z_\theta \rho)k + zh + \frac{1}{2}(u_{\theta\theta}h'h + z_{\theta\theta}(h'h\rho + ch) = \bar{f} + \bar{g} \\ \bar{f}h' + z_\theta \rho h' + zc = a \end{cases}$$

Similarly, if we define $p(\theta, \lambda, t)$ as the probability density function for the Ito processes $[\Theta, \Lambda]$, we can obtain the following evaluation equation for p.

Theorem 5 *The probability density function $p(\theta, \lambda, t)$ for $[\Theta, \Lambda]$ satisfies the following evolution equation,*

$$\begin{aligned} p_t \quad = \quad & -(kp)_\theta - \nabla_\lambda \cdot (hp) + \tfrac{1}{2}(((h'h)p)_{\theta\theta} \\ & +2\nabla_\lambda \cdot ((ch)p)_\theta + \nabla_\lambda \cdot (\nabla_\lambda \cdot (cc')p)), \end{aligned} \tag{59}$$

where $\nabla_\lambda = [\frac{\partial}{\partial \lambda_1}, \cdots, \frac{\partial}{\partial \lambda_{n-1}}]$. And if the starting point of $X(0)$ is at $u(\theta_0) + z(\theta_0)\lambda_0$, the initial condition for (59) is

$$p(\theta, \lambda, 0) = \delta(\theta - \theta_0)\delta(\lambda - \lambda_0), \tag{60}$$

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