Let $\mathbf{v}$ denote a column vector of the nilpotent matrix

$$P_i(A)(A-\lambda_i I)^{n_i-1}$$

where n_i is the so called nilpotency. Theorem 3 in [1] shows that

$$AP_i(A)(A-\lambda_i I)^{n_i-1} = \lambda_i P_i(A)(A-\lambda_i I)^{n_i-1}.$$

which means a column vector $\mathbf{v}$ of the matrix is an eigenvector corresponding to the eigenvalue λ_i. The symbols are explained in [1]. However it is worth noting that

$$P_i(A)(A-\lambda_i I).$$

That is

$$\begin{aligned} P_i(A)(A-\lambda_i I)^{n_i-1} &\neq 0 \\ \text{but } P_i(A)(A-\lambda_i I)^{n_i} &= 0. \end{aligned}$$

We now use the formula for A^t as derived in [2]

$$A^t = \sum_{i=1}^{k} P_i(A)\lambda_i^t \sum_{j=0}^{m_i-1} \frac{\Gamma(t+1)}{j!\Gamma(t-j+1)} (\frac{A-\lambda_i I}{\lambda_i})^j, \text{ assuming } I^t = I \tag{0.1}$$

to compute $A^t\mathbf{v}$ **by,**

$$\begin{aligned} A^t P_s(A)(A-\lambda_s I)^{n_s-1} &= [\sum_{i=1}^{k} P_i(A)\lambda_i^t \sum_{j=0}^{m_i-1} \frac{\Gamma(t+1)}{j!\Gamma(t-j+1)} (\frac{A-\lambda_i I}{\lambda_i})^j][P_s(A)(A-\lambda_s I)^{n_s-1}] \\ &= P_s(A)\lambda_s^t \sum_{j=0}^{m_i-1} \frac{\Gamma(t+1)}{j!\Gamma(t-j+1)} \frac{1}{\lambda_s^j}(A-\lambda_s I)^{j+n_s-1}] \\ \text{by } P_s(A)P_i(A) &= P_s(A)\delta_{is} \end{aligned}$$

Note that by the property of the nilpotency, n_s,

$$P_s(A)(A-\lambda_s I)^{n_s} = 0.$$

Therefore the only nonzero term in the last sum is the leading term corresponding to $j=0$. It follows then

$$A^t P_s(A)(A-\lambda_s I)^{n_s-1} = \lambda_s^t P_s(A)(A-\lambda_s I)^{n_s-1}$$

which proves for each column vector, $\mathbf{v}$, of the matrix $P_s(A)(A-\lambda_s I)^{n_s-1}$ the result you mentioned

$$A^t\mathbf{v} = \lambda_s^t \mathbf{v}.$$

We will use Example 3 in [1] to help explain the notations used here and to illustrate the result.

Let

$$A = \begin{bmatrix} -3 & 5 & -5 \\ 3 & -1 & 3 \\ 8 & -8 & 10 \end{bmatrix}.$$

The characteristic polynomial is

$$p(\lambda) = (\lambda - 2)^3$$

and only one projection matrix is

$$P_1(A) = I$$

so that

$$A^t = 2^t I + t2^{t-1}(A-2I) + \frac{t(t-1)}{2}2^{t-2}(A-2I)^2.$$

More explicitly

$$A^t = 2^{t-1}\begin{bmatrix} 2-5t & 5t & -5t \\ 3t & 2-3t & 3t \\ 8t & -8t & 2+8t \end{bmatrix}.$$

The nilpotent matrix, $P_i(A)(A-\lambda_i I)$ in this case is

$$\begin{aligned} A-2I &= \begin{bmatrix} -5 & 5 & -5 \\ 3 & -3 & 3 \\ 8 & -8 & 8 \end{bmatrix} \text{ and} \\ (A-2I)^2 &= \begin{bmatrix} -5 & 5 & -5 \\ 3 & -3 & 3 \\ 8 & -8 & 8 \end{bmatrix}^2 = 0 \end{aligned}$$

That means $n_i = 2$ and then

$$P_s(A)(A-\lambda_s I)^{n_s-1} = A - 2I$$

an eigenvector is

$$\mathbf{v} = \begin{bmatrix} -5 \\ 3 \\ 8 \end{bmatrix}$$

If we choose $t = 1/2$ in the above formula for A^t, we find

$$\begin{aligned}
A^{1/2} &= \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{1}{2} & \frac{5}{2} & \frac{-5}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \\ 4 & -4 & 6 \end{bmatrix} \\
&= \frac{1}{2\sqrt{2}} \begin{bmatrix} -1 & 5 & -5 \\ 3 & 1 & 3 \\ 8 & -8 & 12 \end{bmatrix} \\
A^{1/2}\mathbf{v} &= \frac{1}{2\sqrt{2}} \begin{bmatrix} -1 & 5 & -5 \\ 3 & 1 & 3 \\ 8 & -8 & 12 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \\ 8 \end{bmatrix} \\
&= \frac{1}{2\sqrt{2}} \begin{bmatrix} -20 \\ 12 \\ 32 \end{bmatrix} = \sqrt{2} \begin{bmatrix} -5 \\ 3 \\ 8 \end{bmatrix} \\
&= \lambda_1^{1/2}\mathbf{v}.
\end{aligned}$$

References

[1] D. V. Ho, *The power & exponential of a matrix,* www.math.gatech.edu/~ho
[2] D. V. Ho, *Real power and logarithm of a matrix,* www.math.gatech.edu/~ho