A Physical Explanation of Reynold's Theorem

Consider a control volume $V$ fixed in space. Let $\phi(\vec{x}, t)$ be the density of some physical quantity of a certain continuum so that the physical quantity in $V$ at time $t$ is given by the integral

$$\phi(t) = \iiint_V \mathcal{F}(\vec{x}, t) \, dV$$

The derivative of $\phi$ is then

$$\phi'(t) = \iiint_V \mathcal{F}_i(\vec{x}, t) \, dV$$

$$\lim_{h \to 0} \frac{1}{h} \{ \iiint_V \mathcal{F}(\vec{x}, t + h) \, dV - \iiint_V \mathcal{F}(\vec{x}, t) \, dV \}, \quad as \ h \to 0.$$ 

Now think of $\phi$ as the population and $V$ as the United States. The difference of the last two integrals is the increment of population of the United States from $t$ to $t + h$ so that $\phi'(t)$ is the pop growth rate of the United States which includes:

(i) the growth rate due to the people who reside in the United State at time $t$

(ii) the growth rate due to immigration, i.e., the influx from its boundaries.

When we deal with a continuum, the rate of growth of type (i) is the material rate of growth often denoted by $D\phi/Dt$. It is this material rate of growth which is concerned in the conservation principles in continuum mechanics. Note then

$$\frac{D\phi}{Dt} = \iiint_V \mathcal{F}_i(\vec{x}, t) \, dV - \iint_S \mathcal{F}(\vec{x}, t) \, \vec{n} \cdot (-\vec{v}) \, dV$$

where $\vec{n}$ is conventionally an outward unit normal to $S$, the boundary of $V$, so here $-\vec{n}$ is used for influx. Applying the divergence theorem on the surface integral, we obtain Reynold's transport theorem

$$\frac{D\phi}{Dt} = \iiint_V \left[ \mathcal{F}_i(\vec{x}, t) + \nabla \cdot (\mathcal{F} \vec{v}) \right] \, dV$$

$$= \iiint_V \left[ \mathcal{F}_i(\vec{x}, t) + (\vec{v} \cdot \nabla) \mathcal{F} + (\nabla \cdot \vec{v}) \mathcal{F} \right] \, dV.$$
Example

Let \( \vec{v} = (0, 0, z) \) and \( \mathcal{F} = xyzt \), a scalar function.
Suppose that at \( t = 0 \), the control volume is bounded by \( x = 0, 1 \); \( y = 0, 1 \)
and \( z = 0, 1 \). Verify Reynold's theorem.

From \( \vec{v} = (0, 0, z) \), we obtain the system of differential equations

\[
R' (t) = (0, 0, z) \quad \text{i.e.} \quad x' = 0, \ y' = 0, \ z' = z
\]

The general solution is \( R_q = (c_1, c_2, c_3 e^t) \), so that all faces of \( V_t \) remains essentially
the same except the face at \( z_0 = 1 \) will move to \( z = z_0 e^t \). Thus

\[
\phi(t) = \int \int \int_{V_t} xyzt \, dV = \int_0^1 \int_0^1 xyt \{ \int_0^z z \, dz \} \, dx \, dy = \frac{1}{8} t e^{2t}
\]

so that

\[
\phi'(t) = \frac{1}{8} (1 + 2t) e^{2t}.
\]

On the other hand,

\[
\begin{align*}
\int \int \int_{V_t} (xyz)_t + (0, 0, z) \cdot \nabla (xyzt) + [ \nabla \cdot (0, 0, z) ] xyzt \, dV &= \\
\int \int \int_{V_t} xyz + z \frac{\partial}{\partial z} (xyzt) + xyzt \, dV &= \\
\int_0^1 \int_0^1 \int_0^z \{ xyz + xyzt + xyzt \} \, dx \, dy \, dz &= \frac{1}{8} (1 + 2t) e^{2t}.
\end{align*}
\]

which indeed agrees with \( \phi'(t) \) so the theorem is verified.