THE POWER AND EXPONENTIAL OF A MATRIX

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Abstract. Two formulas are obtained for the power of a matrix and the matrix exponential..

1. Introduction

In an elementary course on differential equations or difference equations, it is always desirable to have a more elementary way to compute the matrix exponential e^{tA} or the power function A^n of a constant square matrix A. Leonard [1] gave an elegant method of constructing e^{tA} by using the solutions of an initial value problem of a homogeneous matrix equation whose coefficients are just those of the characteristic polynomial of A. Liz [2] simplified it by making the application of the initial conditions in Leonard's method more systematic. Kwapisz [3] gave a natural extension of Leonard's method for A^n in terms of solutions of a difference equation. Several methods for A^n were introduced by Elaydi and Harris [4]. Some of them are similar to that of [3] and others are based on the SN decomposition of A. Harris, Fillmore and Smith [5] extended those methods in [4] for e^{tA}.

In this paper we provide a unified approach to calculate A^n (Theorem 1) , e^{tA} (Theorem 2) and eigenvectors (Theorem 3). Each one of these quantities depends solely on the so-called projection polynomials of A. In Section 2 some of the well-known properties of the projection polynomials are explained here because of their importance in this work. Section 3 starts with Herod's [6] spectral decomposition of A. A formula for A^n is thus computed in the most straightforward fashion. It is then used to sum up the power series for e^{tA} to obtain a familiar result which can be found in [7], [8]. I believe that the formula for A^n is a new result.

Three examples are given in Section 4. They illustrate how to obtain the projection polynomials for a given A and then use them to find A^n, e^{tA} and all eigenvectors.

2. Projection Matrices

As in all the aforementioned papers, we assume that exact eigenvalues of A can be found and we can write down the characteristic polynomial as

$$p(\lambda) = \prod_{i=1}^{k} (\lambda - \lambda_i)^{m_i} \tag{1}$$

where $\lambda_i \neq \lambda_j$ if $i \neq j$ and m_i is the multiplicity of the eigenvalue λ_i. For more complicated matrices, approximations or numerical methods may have to be employed.

In that case the reader is referred to the paper by Moler and Van Loan [9] and the book by Golub and Van Loan [10].

Several polynomials based on $p(\lambda)$ can be introduced. Following Rabenstein [7], we define $a_i(\lambda)$ by expanding

$$\frac{1}{p(\lambda)} = \sum_{i=1}^{k} \frac{a_i(\lambda)}{(\lambda - \lambda_i)^{m_i}}. \tag{2}$$

$b_i(\lambda)$ is defined by omitting the factor $(\lambda - \lambda_i)^{m_i}$ from the characteristic polynomial $p(\lambda)$, that is

$$b_i(\lambda) = \prod_{j \neq i} (\lambda - \lambda_j)^{m_j}. \tag{3}$$

$P_i(\lambda)$ is simply defined as

$$P_i(\lambda) = a_i(\lambda) b_i(\lambda). \tag{4}$$

We show some of the well-known properties of these polynomials. From (2),

$$1 = \sum_{j=1}^{k} \frac{a_j(\lambda)}{(\lambda - \lambda_j)^{m_j}}\, p(\lambda) = \sum_{j=1}^{k} a_j(\lambda) b_j(\lambda) = \sum_{j=1}^{k} P_j(\lambda).$$

When λ is replaced with A, the last identity becomes

$$I = \sum_{i=1}^{k} P_i(A). \tag{5}$$

Proposition 1

$$P_i(A)(A - \lambda_i I)^{m_i} = 0. \tag{6}$$

Proof.

$$\begin{aligned} P_i(A)(A - \lambda_i I)^{m_i} &= a_i(A) b_i(A)(A - \lambda_i I)^{m_i} && \text{by (4)} \\ &= a_i(A) \prod_{j=1}^{k} (A - \lambda_j I)^{m_j} && \text{by (3)} \\ &= a_i(A) p(A) = 0 && \text{by Cayley Hamilton Theorem.} \end{aligned}$$

Proposition 2.
The set $P_i(A)$ are orthogonal in the sense,

$$P_i(A) P_j(A) = \delta_{ij} P_i(A), \tag{7}$$

where δ_{ij} is the Kronecker delta.

Proof.

If $i \neq j$,

$$\begin{aligned} P_i(A)P_j(A) &= P_i(A)a_j(A)b_j(A) = P_i(A)a_j(A)\prod_{k\neq j}(A-\lambda_k I)^{m_k} \\ &= P_i(A)a_j(A)(A-\lambda_i I)^{m_i}\prod_{k\neq i,j}(A-\lambda_k I)^{m_k} = 0 \qquad \text{by (6).} \end{aligned}$$

To complete the proof, we multiply $P_i(A)$ on (5) to find

$$P_i(A) = \sum_{j=1}^{k} P_i(A)P_j(\lambda) = P_i(A)P_i(A).$$

3. The Main Results

Herod [6] has introduced the spectral decomposition of A as

$$A = [\sum_{i=1}^{k} P_i(A)]A = \sum_{i=1}^{k} P_i(A)[\lambda_i I + (A-\lambda_i I)]. \tag{8}$$

Note that for each multiple eigenvalue, there is a nilpotent component $P_i(A)(A-\lambda_i I)$ since

$$[P_i(A)(A-\lambda_i I)]^{m_i} = P_i(A)(A-\lambda_i I)^{m_i} = 0, \qquad \text{by (7), (6).}$$

Using this spectral decomposition, we are able to simplify the calculation of A^n.

Theorem 1 For a nonnegative integer n,

$$A^n = \sum_{i=1}^{k} P_i(A)\sum_{j=0}^{m_i-1}\binom{n}{j}\lambda_i^{n-j}(A-\lambda_i I)^j. \tag{9}$$

Proof.

$$\begin{aligned} A^n &= [\sum_{i=1}^{k} P_i(A)[\lambda_i I + (A-\lambda_i I)]]^n \qquad \text{by (8)} \\ &= \sum_{i=1}^{k} P_i(A)[\lambda_i I + (A-\lambda_i I)]^n \qquad \text{by (7)} \\ &= \sum_{i=1}^{k} P_i(A)\sum_{j=0}^{n}\binom{n}{j}\lambda_i^{n-j}(A-\lambda_i I)^j, \text{ using binomial expansion.} \end{aligned}$$

Note that in view of (6) the last sum will terminate at $j = m_i - 1$ if $n \geq m_i$. If $n < m_i$, the binomial coefficient $\binom{n}{j} = 0$ for $j \geq n + 1$ so that

$$\begin{aligned}\sum_{j=0}^{m_i-1} \binom{n}{j} \lambda_i^{n-j}(A - \lambda_i I)^j &= \sum_{j=0}^{n} \binom{n}{j} \lambda_i^{n-j}(A - \lambda_i I)^j + \sum_{j=n+1}^{m_i-1} \binom{n}{j} \lambda_i^{n-j}(A - \lambda_i I)^j \\ &= \sum_{j=0}^{n} \binom{n}{j} \lambda_i^{n-j}(A - \lambda_i I)^j.\end{aligned}$$

Hence (9) holds for all positive integer n. Notice that for $n = 0$, (9) reduces to the identity $I = I$, if none of the eigenvalues is zero. If an eigenvalue is zero of multiplicity m, equation (9) is valid for $n \geq m$. For the sake of simplicity we shall assume A to be nonsingular.

Theorem 2

$$e^{tA} = \sum_{i=1}^{k} e^{\lambda_i t} P_i(A) \sum_{j=0}^{m_i-1} \frac{t^j}{j!}(A - \lambda_i I)^j \tag{10}$$

where t is a real variable.

Proof.

$$\begin{aligned}e^{tA} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{i=1}^{k} P_i(A) \sum_{j=0}^{m_i-1} \binom{n}{j} \lambda_i^{n-j}(A - \lambda_i I)^j \qquad \text{by (9)} \\ &= \sum_{i=1}^{k} P_i(A) \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{j=0}^{m_i-1} \frac{n!}{j!(n-j)!} \lambda_i^{n-j}(A - \lambda_i I)^j \\ &= \sum_{i=1}^{k} P_i(A) \sum_{j=0}^{m_i-1} (A - \lambda_i I)^j \sum_{n=j}^{\infty} \frac{t^n}{j!(n-j)!} \lambda_i^{n-j} \text{ since } \frac{n!}{(n-j)!} = 0 \text{ for } n < j \\ &= \sum_{i=1}^{k} P_i(A) \sum_{j=0}^{m_i-1} (A - \lambda_i I)^j \sum_{q=0}^{\infty} \frac{t^{q+j}}{j!q!} \lambda_i^q \qquad \text{where } q = n - j \\ &= \sum_{i=1}^{k} P_i(A) \sum_{j=0}^{m_i-1} \frac{t^j}{j!}(A - \lambda_i I)^j \sum_{q=0}^{\infty} \frac{t^q}{q!} \lambda_i^q \\ &= \sum_{i=1}^{k} e^{\lambda_i t} P_i(A) \sum_{j=0}^{m_i-1} \frac{t^j}{j!}(A - \lambda_i I)^j\end{aligned}$$

which agrees with the formula for e^{tA} given in [7, p. 308], [8, p. 510]. This equation should hold for singular A. Should A be singular, we let $B = A + \alpha I$ where α may be chosen that B is not singular. Suppose that μ_i is the i-th eigenvalue of B so that $\mu_i = \lambda_i + \alpha$. It follows that

$$\begin{aligned}
e^{tA} &= e^{-\alpha t}e^{tB} = e^{-\alpha t+\mu_i t}e^{(tB-\mu_i I)} \\
&= e^{t\lambda_i}\sum_{j=1}^{\infty}\frac{t^j}{j!}(B-\mu_i I)^j, \qquad \text{where } B-\mu_i I = A-\lambda_i I \\
&= e^{t\lambda_i}\sum_{j=1}^{\infty}\frac{t^j}{j!}(A-\lambda_i I)^j. \\
\sum_{i=1}^{k}P_i(A)e^{tA} &= \sum_{i=1}^{k}P_i(A)e^{t\lambda_i}\sum_{j=1}^{m_i-1}\frac{t^j}{j!}(A-\lambda_i I)^j, \text{ by } P_i(A)(A-\lambda_i I)^{m_i}=0 \\
e^{tA} &= \sum_{i=1}^{k}P_i(A)e^{t\lambda_i}\sum_{j=1}^{m_i-1}\frac{t^j}{j!}(A-\lambda_i I)^j, \text{ by } \sum_{i=1}^{k}P_i(A)=I,
\end{aligned}$$

which shows that formula (10) is invariant to a change of $B = A + \alpha I$ and it remains valid when A is singular.

4. Eigenvectors

As noted in (6), $P_i(A)(A-\lambda_i I)^{m_i} = 0$. However this nilpotent may become zero at a lower power n_i. Suppose that $P_i(A)(A-\lambda_i I)^{n_i} = 0$ but $P_i(A)(A-\lambda_i I)^{n_i-1} \neq 0$ for $1 \leq n_i \leq m_i$. In other words, $P_i(A)(A-\lambda_i I)^{n_i}$ is a minimum polynomial of A.

Theorem 3.

A nonzero column $\mathbf{v}_r$ of $P_i(A)(A-\lambda_i I)^{n_i-r}$ is a generalized eigenvector of rank r, for $1 \leq r \leq n_i$, associated with λ_i [8, p. 445]. A regular eigenvector is of rank 1.

Proof.

When $r = 1$, we have

$$\begin{aligned}
AP_i(A)(A-\lambda_i I)^{n_i-1} &= \sum_{j=1}^{k}P_j(A)[\lambda_j I + (A-\lambda_j I)]P_i(A)(A-\lambda_i I)^{n_i-1} \text{ by (8)} \\
&= \lambda_i P_i(A)(A-\lambda_i I)^{n_i-1} \qquad\qquad \text{since } (A-\lambda_i I)^{n_i}=0. \\
A\mathbf{v}_1 &= \lambda_i \mathbf{v}_1.
\end{aligned}$$

When $r > 1$, note that

$$\begin{aligned}
P_i(A)(A-\lambda_i I)^{n_i-r} &= (A-\lambda_i I)P_i(A)(A-\lambda_i I)^{n_i-r-1} \\
\mathbf{v}_r &= (A-\lambda_i I)\mathbf{v}_{r+1}.
\end{aligned}$$

Remark Here we note the geometric nature of the projection matrix, $P_i(A)$. If all eigenvalues of A are distinct, the spectral decomposition becomes

$$A = \sum_{i=1}^{k} \lambda_i P_i(A)$$

which illustrates the decomposition of A in its eigenspace. Multiplying $P_j(A)$ on the above equation, we see

$$P_j(A)A = \lambda_j P_j(A), \quad \text{by (7)},$$

which shows that $P_j(A)$ projects (or resolves) A onto the direction of an eigenvector associated with λ_j.

5. Examples

Three examples, with different combinations of eigenvalues, are given to illustrate the simplicity of our approach. After the eigenvalues are found, the projection polynomials are computed and then substituted into (9) for A^n and into (10) for e^{tA}. The eigenvectors can be selected from these polynomials according to Theorem 3. Needless to say, the approach is especially advantageous if all the three quantities are to be computed for a given A. The examples will highlight the fact that the calculations are simple and elementary, except some messy algebraic manipulations in the last step to reach the explicit matrix for A^n or e^{tA}. In the end of the first example we see that none of these messy manipulations will be encountered in obtaining the asymptotic behavior of these functions.

Example 1.

Let $A = \begin{bmatrix} -\frac{3}{4} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -\frac{3}{2} & -1 \end{bmatrix}$.

The characteristic polynomial is $p(\lambda) = (\lambda + \frac{3}{4})(\lambda + \frac{1}{2})(\lambda + \frac{1}{4})$ and thus $\lambda_1 = -\frac{3}{4}$, $\lambda_2 = -\frac{1}{2}$, $\lambda_3 = -\frac{1}{4}$, $m_1 = m_2 = m_3 = 1$.

By partial fractions, we find

$$\begin{aligned} \frac{1}{p(\lambda)} &= \frac{1}{(\lambda+3/4)(\lambda+1/2)(\lambda+1/4)} = \frac{8}{\lambda+3/4} - \frac{16}{\lambda+1/2} + \frac{8}{\lambda+1/4}\,, \qquad \text{and thus} \\ a_1(\lambda) &= 8,\ a_2(\lambda) = -16,\ a_3(\lambda) = 8 \qquad \text{by (2)} \\ b_1(\lambda) &= (\lambda+\frac{1}{2})(\lambda+\frac{1}{4}),\ b_2(\lambda) = \left(\lambda+\frac{1}{4}\right)(\lambda+\frac{3}{4}),\ b_3(\lambda) = (\lambda+\frac{3}{4})(\lambda+\frac{1}{2}) \qquad \text{by (3)} \\ P_1(\lambda) &= (2\lambda+1)(4\lambda+1),\ P_2(\lambda) = -(4\lambda+1)(4\lambda+3),\ P_3(\lambda) = (4\lambda+3)(2\lambda+1) \qquad \text{by(4)}. \end{aligned}$$

Substituting λ_i, $P_i(A)$ into (9), we get

$$\begin{aligned} A^n &= \lambda_1^n P_1(A) + \lambda_2^n P_2(A) + \lambda_3^n P_3(A) \\ &= (-\frac{3}{4})^n (2A+I)(4A+I) - (-\frac{1}{2})^n (4A+I)(4A+3I) + (-\frac{1}{4})^n (4A+3I)(2A+I) \\ &= \begin{matrix} \left(-\frac{3}{4}\right)^n & 0 & 0 \\ 2\left(-\frac{1}{2}\right)^n - 2\left(-\frac{1}{4}\right)^n & -2\left(-\frac{1}{2}\right)^n + 3\left(-\frac{1}{4}\right)^n & -\left(-\frac{1}{2}\right)^n + \left(-\frac{1}{4}\right)^n \\ 2\left(-\frac{3}{4}\right)^n - 6\left(-\frac{1}{2}\right)^n + 4\left(-\frac{1}{4}\right)^n & 6\left(-\frac{1}{2}\right)^n - 6\left(-\frac{1}{4}\right)^n & 3\left(-\frac{1}{2}\right)^n - 2\left(-\frac{1}{4}\right)^n \end{matrix}. \end{aligned}$$

Similarly we compute e^{tA}, using (10),

$$\begin{aligned} e^{tA} &= e^{-3t/4} P_1(A) + e^{-t/2} P_2(A) + e^{-t/4} P_3(A) \\ &= e^{-3t/4}(2A+I)(4A+I) - e^{-t/2}(4A+I)(4A+3I) + e^{-t/4}(4A+3I)(2A+I) \\ &= \begin{bmatrix} e^{-\frac{3}{4}t} & 0 & 0 \\ 2e^{-\frac{1}{2}t} - 2e^{-\frac{1}{4}t} & -2e^{-\frac{1}{2}t} + 3e^{-\frac{1}{4}t} & -e^{-\frac{1}{2}t} + e^{-\frac{1}{4}t} \\ 2e^{-\frac{3}{4}t} - 6e^{-\frac{1}{2}t} + 4e^{-\frac{1}{4}t} & 6e^{-\frac{1}{2}t} - 6e^{-\frac{1}{4}t} & 3e^{-\frac{1}{2}t} - 2e^{-\frac{1}{4}t} \end{bmatrix}. \end{aligned}$$

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Since the eigenvalues are distinct, three eigenvectors , $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ are found from the projection matrices

$P_1(A) = (2A+I)\,(4A+I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$,

$P_2(A) = -(4A+I)(4A+3I) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -2 & -1 \\ -6 & 6 & 3 \end{bmatrix}$,

$P_3(A) = (4A+3I)(2A+I) = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 3 & 1 \\ 4 & -6 & -2 \end{bmatrix}$, respectively.

For the asymptotic behavior, we need only to note,

$$A^n = (-\frac{3}{4})^n (2A+I)(4A+I) - (-\frac{1}{2})^n (4A+I)(4A+3I) + (-\frac{1}{4})^n (4A+3I)(2A+I) \to 0 \qquad as\ n \to \infty.$$

and

$$\begin{aligned} e^{tA} &= e^{-3t/4}(2A+I)(4A+I) - e^{-t/2}(4A+I)(4A+3I) + e^{-t/4}(4A+3I)(2A+I) \\ &\to 0, \qquad as\ t \to \infty. \end{aligned}$$

Example 2.

Let $A=\begin{bmatrix} 2 & -2 & 1 \\ 1 & -1 & 1 \\ -3 & 2 & -2 \end{bmatrix}$. We find $p(\lambda)=(\lambda+1)^2(\lambda-1)$ so that $\lambda_1=-1$, $m_1=2$, $\lambda_2=1$, $m_2=1$ and

$$\begin{aligned}
\frac{1}{p(\lambda)} &= \frac{1}{(\lambda+1)^2(\lambda-1)}=-\frac{1}{4}\frac{\lambda+3}{(\lambda+1)^2}+\frac{1}{4\,(\lambda-1)} \\
a_1(\lambda) &= -(\lambda+3)/4, \quad a_2(\lambda)=1/4 \\
b_1(\lambda) &= \lambda-1, \quad b_2(\lambda)=(\lambda+1)^2 \\
P_1(\lambda) &= -\frac{1}{4}(\lambda+3)(\lambda-1), \quad P_2(\lambda)=\frac{1}{4}(\lambda+1)^2.
\end{aligned}$$

We obtain from (9),

$$\begin{aligned}
A^n &= P_1(A)[\lambda_1^n I+n\lambda_1^{n-1}(A-\lambda_1 I)]+P_2(A)\lambda_2^n \\
&= -\frac{1}{4}(A+3I)(A-I)((-1)^n I+n(-1)^{n-1}(A+I))+\frac{1}{4}(A+I)^2 \\
&= \begin{bmatrix} -(-1)^n n+1 & (-1)^n-1 & -(-1)^n n \\ -(-1)^n n & (-1)^n & -(-1)^n n \\ (-1)^n n+(-1)^n-1 & -(-1)^n+1 & (-1)^n n+(-1)^n \end{bmatrix}
\end{aligned}$$

and from (10),

$$\begin{aligned}
e^{tA} &= e^{-t}P_1(A)(I+t(A+I))+e^t P_2(A) \\
&= -\frac{1}{4}(A+3I)(A-I)(I+t(A+I))e^{-t}+\frac{1}{4}(A+I)^2 e^t \\
&= \begin{bmatrix} e^{-t}t+e^t & e^{-t}-e^t & e^{-t}t \\ e^{-t}t & e^{-t} & e^{-t}t \\ e^{-t}(1-t)-e^t & -e^{-t}+e^t & e^{-t}(1-t) \end{bmatrix}.
\end{aligned}$$

Associated with $\lambda_1=-1$, for which $m_1=2$,

$$\begin{aligned}
P_1(A) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}, \quad P_1(A)(A-\lambda_1 I)=\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix} \\
P_1(A)(A+I)^2 &= 0 \quad \text{by (6)}
\end{aligned}$$

so $n_1 = 2$. One regular eigenvector is $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and a rank 2 generalized eigenvector is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. For λ_2, a distinct eigenvalue, $n_2 = m_2 = 1$. We have therefore a regular eigenvector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ from $P_2(A) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}$.

Example 3.

$A = \begin{bmatrix} -3 & 5 & -5 \\ 3 & -1 & 3 \\ 8 & -8 & 10 \end{bmatrix}$, $p(\lambda) = (\lambda - 2)^3$, $\lambda_1 = 2$, $m_1 = 3$, $P_1(A) = I$. Then

$$\begin{aligned} A^n &= 2^n I + n2^{n-1}(A - 2I) + \frac{n(n-1)}{2} 2^{n-2}(A - 2I)^2 \\ &= 2^{n-1} \begin{bmatrix} 2-5n & 5n & -5n \\ 3n & 2-3n & 3n \\ 8n & -8n & 2+8n \end{bmatrix} \qquad \text{by (9)}. \\ e^{tA} &= e^{2t}(I + t(A - 2I) + \frac{t^2}{2}(A - 2I)^2) \\ &= e^{2t} \begin{bmatrix} 1-5t & 5t & -5t \\ 3t & 1-3t & 3t \\ 8t & -8t & 1+8t \end{bmatrix} \qquad \text{by (10)}. \\ P_1(A)(A - 2I) &= \begin{bmatrix} -5 & 5 & -5 \\ 3 & -3 & 3 \\ 8 & -8 & 8 \end{bmatrix}, \quad P_1(A)(A - 2I)^2 = 0 \end{aligned}$$

Therefore $n_1 = 2$. From $P_1(A)(A - 2I)$, we find an eigenvector $\begin{bmatrix} 5 \\ -3 \\ -8 \end{bmatrix}$. Since $P_1(A)(A - \lambda_1 I)^{n_1 - 2} = I$, we may choose $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ as two generalized eigenvectors of rank 2.

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