Highest-weight vectors of the moonshine module with non-zero graded trace

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1 Introduction

Any element of a vertex operator algebra (VOA) generates a graded trace (one-point correlation function). For holomorphic VOAs, Zhu proves that these graded traces are modular forms [Zhu96]. In their paper, *Monstrous moonshine of higher weight* [DM00], Dong and Mason describe the space of graded traces generated by the moonshine module, $V^\natural$. In particular, they find that it contains all cusp forms. Additionally, they explicitly compute the graded traces of a family of highest-weight vectors, achieving non-zero cusp forms of weights congruent to zero modulo four, and conjecture that every cusp form can be achieved as the graded trace of a (monster-invariant) *highest-weight* vector of $V^\natural$. In this paper we compute the graded traces of another family of highest-weight vectors obtaining non-zero cusp forms of weight congruent to two modulo four.

2 Notation and Definitions

We use the definition of vertex operator algebra given in *On Axiomatic Approaches to Vertex Operator Algebras and Modules*[FHL93] and also adopt most of their notation. A VOA is specified $(V, Y, \mathbf{1}, \omega)$; V is the underlying vector space, $Y(v,z) = \sum_{n\in\mathbb{Z}} v(n)z^{-n-1}$ is the vertex operator associated with v, $\mathbf{1}$ is the vacuum vector, and ω is the conformal vector. We denote the central charge c and let $V = \oplus_{n\geq n_0} V_n$ be the decomposition of V into homogeneous spaces. Vertex operator algebras are, by definition, representations of the Virasoro algebra via $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$. By a *highest-weight vector* v of V we mean a highest-weight vector of this representation, i.e., $L(n)v = 0$ for all $n > 0$.

One-point correlation functions are defined by Zhu [Zhu96]. We call them graded traces, because the term is shorter and more algebraic. Let $v \in V_k$ and define $o(v) = v(k-1)$. The linear map $o(v)$ preserves the grading of V and we define the *graded trace* to be

$$Z_V(v, q) = \mathrm{tr}_V\, o(v) q^{L(0)-c/24} := q^{-c/24} \sum_{n\in\mathbb{Z}} \mathrm{tr}\,\big|_{V_n} o(v) q^n. \tag{1}$$

We also use the notation $Z_V(v, \tau)$; τ is related to q via $q = e^{2\pi i\tau}$.

The moonshine module, $V^\natural$, is a VOA constructed using the Leech lattice VOA ,V_Λ. In our calculations of graded traces we will need explicit formulas for the vertex operators of V_Λ. To this end we summarize the definition of lattice VOA below. (For more details, see [Don93].) Further aspects of the definition of the moonshine module will be presented in the text as needed. (For the complete construction of $V^\natural$, see [FLM88].)

Let L be a positive-definite even integral lattice with inner product (,) and let $\mathfrak{h} := L \otimes_{\mathbb{Z}} \mathbb{C}$. The corresponding Heisenberg Lie algebra is $\hat{\mathfrak{h}}_{\mathbb{Z}} := \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$, and $M(1) := U(\hat{\mathfrak{h}}_{\mathbb{Z}}) \otimes_{U(\mathfrak{h}\otimes\mathbb{C}[t]\oplus\mathbb{C}c)} M$ is the irreducible $\hat{\mathfrak{h}}_{\mathbb{Z}}$-module induced from the one-dimensional $\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}c$-module M, in which elements of $\mathfrak{h} \otimes \mathbb{C}[t]$ act as 0, and c acts as the identity. Let $\{\alpha_1, \alpha_2, ..., \alpha_m\}$ be a basis for $\mathfrak{h}$, and denote the element $\alpha \otimes t^n$ of $\hat{\mathfrak{h}}_{\mathbb{Z}}$ by $\alpha(n)$. Polynomials in the variables $\alpha_i(-n)$, where $1 \le i \le m$ and $n \ge 1$, form a basis of $M(1)$.

Since L is a free abelian group of finite rank, we can define a $\mathbb{Z}$-bilinear cocycle $\epsilon : L \times L \to \mathbb{Z}/2\mathbb{Z}$ such that $\epsilon(\alpha, \beta) - \epsilon(\beta, \alpha) \equiv (\alpha, \beta) \mathrm{mod}\ 2$. Let $\mathbb{C}\{L\}$ be the twisted group algebra with basis $\{e^\beta | \beta \in L\}$ and multiplication $e^\alpha e^\beta = (-1)^{\epsilon(\alpha,\beta)} e^{\alpha+\beta}$. The lattice VOA, V_L, is defined to be $M(1) \otimes \mathbb{C}\{L\}$. It is linearly isomorphic to $S(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes \mathbb{C}[L]$.

Let $v = \alpha_1(-n_1)\alpha_2(-n_2)\dots\alpha_k(-n_k) \otimes e^\beta$ be an element of V_L.

$$\begin{aligned} Y(v, z) = {}_\circ^\circ \bigg(\frac{1}{(n_1-1)!} \Big(\frac{\partial}{\partial z}\Big)^{n_1-1} \alpha_1(z) \bigg) \dots \\ \bigg(\frac{1}{(n_k-1)!} \Big(\frac{\partial}{\partial z}\Big)^{n_k-1} \alpha_k(z) \bigg) E^-(-\beta, z) E^+(-\beta, z) e^\beta z^\beta {}_\circ^\circ \end{aligned} \tag{2}$$

where

$$\alpha(z) = \sum_{n\in\mathbb{Z}} \alpha(n) z^{-n-1},$$
$$E^\pm(\beta, z) = \exp\bigg(\sum_{n\in\mathbb{N}} \frac{\beta(\pm n)}{\pm n} z^{\mp n} \bigg).$$

The actions of $\alpha(n)$ on $u \otimes e^\gamma \in V_L$ are

$$\alpha(n) \cdot u \otimes e^\gamma = \begin{cases} (\alpha(n)u) \otimes e^\gamma & \text{if } n < 0, \\ n \frac{\partial u}{\partial(\alpha(-n))} \otimes e^\gamma & \text{if } n > 0, \\ (\alpha, \gamma) u \otimes e^\gamma & \text{if } n = 0. \end{cases} \tag{3}$$

The actions of e^β and z^β are

$$e^\beta \cdot u \otimes e^\gamma = (-1)^{\epsilon(\beta,\gamma)} u \otimes e^{\beta+\gamma}, \tag{4}$$
$$z^\beta \cdot u \otimes e^\gamma = u \otimes e^\gamma z^{(\beta,\gamma)}.$$

The normal ordering $\,^\circ_\circ \cdots \,^\circ_\circ$ of the operators in $Y(v, z)$ indicates that $\alpha(n)$ with positive n acts before, i.e., to the right of, $\alpha(n)$ with negative n and that $\alpha(0)$ and z^β act before e^β.

3 Highest-Weight Vectors

The moonshine module is the direct sum of two spaces. The first is a sub-VOA of the Leech Lattice VOA, V_Λ. It is denoted V_Λ^+. The second is a twisted module for V_Λ. It is denoted $(V_\Lambda^T)^+$. Because we will be directly computing graded traces, we limit our search for highest-weight vectors of $V^\natural$ to elements of V_Λ^+. The vertex operators associated with these elements are known explicitly, unlike those of $(V_\Lambda^T)^+$.

The conformal vector ω of $V^\natural$ is the same as that of V_Λ, namely $\sum_{i=1}^{24} \alpha_i(-1)^2$, where $\{\alpha_i \mid 1 \leq i \leq 24\}$ is an orthonormal basis for $\mathfrak{h} = \Lambda \otimes_\mathbb{Z} \mathbb{C}$. The set of highest-weight vectors of $V^\natural$ in V_Λ^+ contains exactly those highest-weight vectors of V_Λ that lie in V_Λ^+. However, the graded traces of these vectors on the spaces $V^\natural$ and V_Λ differ, and there are highest-weight vectors with non-zero trace on $V^\natural$ and zero trace on V_Λ. In fact, it is not hard to show that any highest-weight vector of V_Λ that is not in $M(1)$ has a graded trace on V_Λ equal to zero. This is not necessarily true of its graded trace on $V^\natural$. Dong and Mason compute the graded trace of $1 \otimes (e^\lambda + e^{-\lambda})$ on $V^\natural$ for all $0 \neq \lambda \in 2\Lambda$ and find the graded trace to be non-zero for $(\lambda, \lambda) \geq 24$ [DM00].

First we examine V_Λ^+ more closely. It is the sub-VOA of $V_\Lambda = M(1) \otimes \mathbb{C}\{\Lambda\}$ fixed by the VOA automorphism t, which is defined by $t(e^\lambda) = e^{-\lambda}$ for $e^\lambda \in \mathbb{C}\{\Lambda\}$ and

$$t\big(\alpha_1(-n_1)\alpha_2(-n_2)...\alpha_k(-n_k)\big) = (-1)^k \big(\alpha_1(-n_1)\alpha_2(-n_2)...\alpha_k(-n_k)\big) \tag{5}$$

for $\alpha_1(-n_1)\alpha_2(-n_2)...\alpha_k(-n_k) \in M(1)$. For β in Λ, define $V(\beta)$ to be $M(1) \otimes (\mathbb{C}e^\beta + \mathbb{C}e^{-\beta})$ and $V(\beta)^+$ to be those elements of $V(\beta)$ fixed by t. Clearly, we have the following vector-space direct-sum decomposition;

$$V_\Lambda^+ = \bigoplus_{\beta \in \Lambda/<\pm 1>} V(\beta)^+,$$

in which $\bigoplus_{\beta\in\Lambda/<\pm1>}$ means that the sum is taken over exactly one element from all pairs $\pm\beta$.

Lemma 3.1. *Let $v \in V(\beta)^+$. There exists a unique $u \in M(1)$ such that*

$$v = u \otimes e^\beta + t(u) \otimes e^{-\beta}.$$

Proof. Let $v \in V(\beta)^+$ and let $\{u_\alpha | \alpha \in A\}$ be a basis for $M(1)$. As $v \in V(\beta)$, we can write

$$v = \sum_{\alpha\in A} c_\alpha \big(u_\alpha \otimes e^\beta\big) + \sum_{\alpha\in A} d_\alpha \big(u_\alpha \otimes e^{-\beta}\big),$$

for some unique $c_\alpha, d_\alpha \in \mathbb{C}$. Define $u := \sum_{\alpha\in A} c_\alpha u_\alpha$ and $u' := \sum_{\alpha\in A} d_\alpha u_\alpha$. As $v \in V_\Lambda^+$, $v = t(v)$, i.e.,

$$u \otimes e^\beta + u' \otimes e^{-\beta} = t(u') \otimes e^\beta + t(u) \otimes e^{-\beta}.$$

The uniqueness of u' implies that $t(u) = u'$ and the result follows. □

We will also find it useful to know how the vertex operators act on the subspaces $V(\beta)^+$.

Lemma 3.2. *Let $\beta, \gamma \in \Lambda$ and $v \in V(\gamma)^+$. Then*

$$Y(v, z) : V(\beta)^+ \to \big(V(\beta+\gamma)^+ \cup V(\beta-\gamma)^+\big)\,[[z, z^{-1}]].$$

Proof. Let $v \in V(\gamma)^+$ and $w \in V(\beta)^+$. For all $n \in \mathbb{Z}$, $v(n)w \in V(\beta+\gamma) \cup V(\beta-\gamma)$. To see this examine the vertex operator for V_Λ^+. It is the restriction of the vertex operator for V_Λ given in equation (2). (In particular, see equation (4).) Furthermore, since t is a VOA automorphism, $v(n)w \in V_\Lambda^+$. Hence $v(n)w \in (V(\beta+\gamma) \cup V(\beta-\gamma)) \cap V_\Lambda^+ = V(\beta+\gamma)^+ \cup V(\beta-\gamma)^+$. □

Let $v \in V_\Lambda^+$ be a highest-weight vector with decomposition $v = \sum_{\beta\in\Lambda/<\pm1>} v_\beta$ such that $v_\beta \in V(\beta)^+$. Since v is a highest-weight vector, for all $n > 0$

$$0 = L(n)v = \sum_{\beta\in\Lambda/<\pm1>} L(n)v_\beta.$$

The conformal vector of $V^\natural$ is in $V(0)^+$. So by Lemma 3.2, $L(n)v_\beta \in V(\beta)^+$. The uniqueness of the decomposition $0 = \sum_{\beta\in\Lambda/<\pm1>} 0$ implies that $L(n)v_\beta = 0$

for all $n > 0$. Therefore v_β is a highest-weight vector. Thus finding the highest-weight vectors in all the $V(\beta)^+$s would suffice to describe the highest-weight vectors in V_Λ^+.

Define $V(\beta)_n^+ := \{u\otimes e^\beta + t(u)\otimes e^{-\beta} | \operatorname{wt} u = n\}$. Clearly $V(\beta)^+ = \bigoplus_{n\geq 0} V(\beta)_n^+$, and as $\operatorname{wt}(L(m)v) = \operatorname{wt} v - m$;

$$L(m) : V(\beta)_n^+ \to \begin{cases} V(\beta)_{n-m}^+ & \text{if } m \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

So if $v \in V(\beta)^+$ is a highest-weight vector and $v = \sum_{n\geq 0} v_n$ with $v_n \in V(\beta)_n^+$, then v_n is a highest-weight vector for all n. Therefore, to find it would suffice to find the highest-weight vectors in all the $V(\beta)_n^+$s.

Our biggest challenge is not finding highest-weight vectors but finding highest-weight vectors with non-zero graded trace on $V^\natural$. The graded trace on $V^\natural$ is the sum of the graded traces on V_Λ^+ and $(V_\Lambda^T)^+$. To increase the odds of getting a non-zero trace on $V^\natural$, we assume that the graded trace on V_Λ^+ is non-zero.

Proposition 3.3. *If $v \in V(\lambda)^+$ and $Z_{V_\Lambda^+}(v, q) \neq 0$ then $\lambda \in 2\Lambda$. Furthermore, if $\lambda \neq 0$, then*

$$Z_{V_\Lambda^+}(v, q) = q^{(\lambda,\lambda)/8} \sum_{n\geq 0} \operatorname{tr}|_{V(\lambda/2)_n^+} o(v) q^n.$$

Proof. The VOA V_Λ^+ has central charge 24, and the weight of the elements of $V(\beta)_n^+$ is $(\beta, \beta)/2 + n$. Thus the graded trace, equation (1), on V_Λ^+ is

$$Z_{V_\Lambda^+}(v, q) = q^{(\beta,\beta)/2-1} \sum_{\beta\in\Lambda/<\pm 1>} \sum_{n\geq 0} \operatorname{tr}|_{V(\beta)_n^+} o(v) q^n. \tag{6}$$

By supposition, $Z_{V_\Lambda^+}(v, q) \neq 0$. Hence $\operatorname{tr}|_{V(\beta)_n^+} o(v) \neq 0$ for some $\beta \in \Lambda$. Therefore the intersection of $o(v)\,(V(\beta)^+)$ and $V(\beta)^+$ must be non-trivial. By Lemma 3.2, this implies that $\lambda = 0$ or $\lambda = \pm 2\beta$. In both cases $\lambda \in 2\Lambda$. Now assume that $\lambda \neq 0$. The only $V(\beta)^+$ which contributes to $Z_{V_\Lambda^+}(v, q)$ is $V(\lambda/2)^+$. The second assertion follows from equation (6). □

In this paper we treat the $\lambda \neq 0$ case; the calculation for $\lambda = 0$ is quite different. For each $n \geq 0$, we would like to find the graded trace on $V^\natural$ of highest-weight vectors in $S_n := \{v \in V(\lambda)_n^+ | \lambda \neq 0, \lambda \in 2\Lambda\}$. The weight of an element of S_n is $(\lambda, \lambda)/2 + n$. So if n is odd, the weight of all $v \in S_n$ is also odd. The graded trace of a highest-weight vector v on $V^\natural$ is a modular form of level one and weight equal to the weight of v [Zhu96]. It is known that the only such form with odd weight is zero. (See, for example [Ser73].) Therefore only even n can give non-zero graded traces. Dong and Mason did the $n = 0$ case [DM00]. The next case of interest is $n = 2$.

Henceforth fix a non-zero $\lambda \in 2\Lambda$ and an orthonormal basis $\{\alpha_1, \alpha_2, \ldots, \alpha_{24}\}$ of $\mathfrak{h}$ such that $\lambda = \ell\alpha_1$. The subspace of $M(1)$ composed of weight-two elements has basis

$$\{\alpha_i(-1)^2, \alpha_i(-1)\alpha_j(-1), \alpha_i(-2) | i \neq j, 1 \leq i, j \leq 24\}.$$

Substituting the elements of this set for u in the expression $u \otimes e^{\lambda} + t(u) \otimes e^{-\lambda}$ produces a basis of $V(\lambda)_2^+$.

Using the Virasoro relations, $[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}c$, we can show that $L(1) \cdot v = 0$ and $L(2) \cdot v = 0$ imply that $L(n) \cdot v = 0$ for all $n \geq 3$, i.e., that v is a highest weight vector.

Direct computation using $\omega = \sum_{i=1}^{24} \alpha_i(-1)^2$ and $Y_{V^\natural}(\omega, z)v = Y_{V_\Lambda}(\omega, z)v = \sum_{n \in \mathbb{Z}} L(n)vz^{-n-2}$ for $v \in V_\Lambda^+$ gives

$$\begin{aligned} L(1) &= \sum_{i=1}^{24} \sum_{n \geq 1} \alpha_i(1-n)\alpha_i(n), \\ L(2) &= \sum_{i=1}^{24} \Big(\frac{1}{2}\alpha_i(1)^2 + \sum_{n \geq 2} \alpha_i(2-n)\alpha_i(n) \Big). \end{aligned}$$

We know that $L(1)$ takes $V(\lambda)_2^+$ to $V(\lambda)_1^+$ and $L(2)$ takes $V(\lambda)_2^+$ to $V(\lambda)_0^+$. Since $V(\lambda)_1^+ \cap V(\lambda)_0^+ = 0$, we can define

$$\begin{aligned} L(1) + L(2) : V(\lambda)_2^+ &\to V(\lambda)_1^+ \oplus V(\lambda)_0^+ \\ \big(L(1) + L(2)\big)v &\mapsto L(1)v + L(2)v. \end{aligned}$$

The kernel of $L(1) + L(2)$ equals the intersection of the kernels of $L(1)$ and $L(2)$, which is the set of highest weight vectors in $V(\lambda)_2^+$.

Let $2 \leq i, j \leq 24$ with $i \neq j$. The actions of $L(1) + L(2)$ on our basis elements are

$$\begin{aligned} \alpha_1(-1)^2 \otimes (e^{\lambda} + e^{-\lambda}) &\mapsto 2\ell\alpha_1(-1) \otimes (e^{\lambda} - e^{-\lambda}) + 1 \otimes (e^{\lambda} + e^{-\lambda}), \\ \alpha_i(-1)^2 \otimes (e^{\lambda} + e^{-\lambda}) &\mapsto 1 \otimes (e^{\lambda} + e^{-\lambda}), \\ \alpha_1(-1)\alpha_i(-1) \otimes (e^{\lambda} + e^{-\lambda}) &\mapsto \ell\alpha_i(-1) \otimes (e^{\lambda} - e^{-\lambda}), \\ \alpha_i(-1)\alpha_j(-1) \otimes (e^{\lambda} + e^{-\lambda}) &\mapsto 0, \\ \alpha_1(-2) \otimes (e^{\lambda} - e^{-\lambda}) &\mapsto 2\alpha_1(-1) \otimes (e^{\lambda} - e^{-\lambda}) + 2\ell \otimes (e^{\lambda} + e^{-\lambda}), \\ \alpha_i(-2) \otimes (e^{\lambda} - e^{-\lambda}) &\mapsto 2\alpha_i(-1) \otimes (e^{\lambda} - e^{-\lambda}). \end{aligned}$$

Finding the null space of $L(1) + L(2)$ yields the following proposition.

Proposition 3.4. *The following set is a basis for the space of highest weight-vectors in* $V(\lambda)_2^+, \lambda \neq 0$.

$$\big\{\alpha_i(-1)\alpha_j(-1)\otimes(e^{\lambda}+e^{-\lambda}), 2\alpha_1(-1)\alpha_2(-1)\otimes(e^{\lambda}+e^{-\lambda})-\ell\alpha_i(-2)\otimes(e^{\lambda}-e^{-\lambda}), \\ -\ell\alpha_1(-2)\otimes(e^{\lambda}+e^{-\lambda})+\big(2\ell^2-1\big)\,\alpha_i^2\otimes(e^{\lambda}+e^{-\lambda})+\alpha_1(-1)^2\otimes(e^{\lambda}+e^{-\lambda}) \\ \big|2\leq i,j\leq 24, i\neq j\big\}$$

The first two types of vector listed in the proposition have graded trace equal to zero. That the last type does not is an implication of our main theorem.

4 Main Theorem

Recall that Λ is the Leech lattice, λ is a non-zero element of 2Λ and $\{\alpha_1, \alpha_2, \cdots, \alpha_{24}\}$ is an orthonormal basis of $\mathfrak{h}$ such that $\lambda = \ell\alpha_1$ for some $\ell \in \mathbb{C}$. Fix a basis element α_i, $i \neq 1$ and define the following element of $V^\natural$;

$$v(\lambda) := -\ell\left(\alpha_1(-2)\otimes(e^{\lambda}-e^{-\lambda})\right) \\ +(2\ell^2-1)\left(\alpha_i(-1)^2\otimes(e^{\lambda}+e^{-\lambda})\right)+\alpha_1(-1)^2\otimes(e^{\lambda}+e^{-\lambda}).$$

Theorem 4.1. *Let* $0 \neq \lambda \in 2\Lambda$ *and let* $v(\lambda)$ *be the element of* $V^\natural$ *defined above. Then* $v(\lambda)$ *is a highest-weight vector of weight* $k = (\lambda,\lambda)/2 + 2$ *and*

$$Z_{V^\natural}(v(\lambda),\tau) = \frac{(\lambda,\lambda)}{8}\eta(\tau)^{12}\Bigg\{2\big(E_2(\tau)-2E_2(2\tau)\big)\left(\frac{\Theta_1(\tau)}{2}\right)^{(\lambda,\lambda)-12} \\ +\big(-E_2(\tau/2)+2E_2(\tau)\big)\left(\frac{\Theta_2(\tau)}{2}\right)^{(\lambda,\lambda)-12} \\ +\big(-E_2(\tau/2)+4E_2(\tau)-4E_2(2\tau)\big)\left(\frac{\Theta_3(\tau)}{2}\right)^{(\lambda,\lambda)-12}\Bigg\}.$$

Remark 4.2. *If* $\lambda = 0$ *or* $\lambda \notin 2\Lambda$ *then* $v(\lambda)$ *is still a highest-weight vector; however, in those cases* $Z_{V^\natural}(v(\lambda),\tau) = 0$. *For* $\lambda \in 2\Lambda$, *if* $(\lambda,\lambda) \geq 32$, *then* $Z_{V^\natural}(v(\lambda), z)$ *is non-zero.*

Proposition 3.4 asserts that $v(\lambda)$ is a highest weight vector. Computing its weight using the $L(0)$ operator is straightforward. To compute the graded trace of $v(\lambda)$ on $V^\natural = V_\Lambda^+ \oplus (V_\Lambda^T)^+$, we start with the trace on V_Λ^+.

By proposition 3.3, to calculate $Z_{V_\Lambda^+}(v,q)$ we need only consider the trace of $o(v(\lambda))$ on $V(\beta)^+$, where $\beta = \lambda/2$.

Let $u(\beta) = u \otimes e^{\beta} + t(u) \otimes e^{-\beta} \in V(\beta)^+$ and

$$v := -\ell\alpha_1(-2) + \left(2\ell^2 - 1\right) \alpha_i(-1)^2 + \alpha_1(-1)^2. \tag{7}$$

We can write $v(\lambda)$ as $v \otimes e^{\lambda} + t(v) \otimes e^{-\lambda}$. Only the terms of $Y(v(\lambda), z)u(\beta)$ in $V(\beta)^+$ can contribute to the trace. These are

$$Y(v \otimes e^{\lambda}, z)t(u) \otimes e^{-\beta} + Y(t(v) \otimes e^{-\lambda}, z)u \otimes e^{\beta}.$$

Let $E^-(\mp\lambda, z)E^+(\mp\lambda, z) = \sum_{n\in\mathbb{Z}} E^{\mp}(n)z^{-n}$. The vertex operators on V_Λ^+ are the restrictions of those on V_Λ given in section 2. Bearing in mind that $2\beta = \lambda = \ell\alpha_1$, we calculate:

$$\begin{aligned} &Y(\pm\alpha_1(-2) \otimes e^{\pm\lambda}, z)u \otimes e^{\mp\beta} \\ &\quad = \pm {}^\circ_\circ \sum_{m,n\in\mathbb{Z}} (-m-1)\alpha_1(m)E^{\mp}(n)z^{-m-n-2}e^{\pm\lambda}z^{\pm\lambda} {}^\circ_\circ u \otimes e^{\mp\beta}, \\ &\quad = z^{-\ell^2/2} {}^\circ_\circ \Big((\mp\alpha_1, \mp\beta)z^{-2} \pm \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} (-m-1)\alpha_1(m)z^{-m-2} \Big) \\ &\qquad \cdot \sum_{n\in\mathbb{Z}} E^{\mp}(n)z^{-n} {}^\circ_\circ u \otimes e^{\pm\beta}, \end{aligned}$$

$$\begin{aligned} &Y(\alpha_k(-1)^2 \otimes e^{\pm\lambda}, z)u \otimes e^{\mp\beta} \\ &\quad = {}^\circ_\circ \sum_{j,m,n\in\mathbb{Z}} \alpha_k(j)\alpha_k(m)E^{\mp}(n)z^{-j-m-n-2}e^{\pm\lambda}z^{\pm\lambda} {}^\circ_\circ u \otimes e^{\mp\beta}, \\ &\quad = z^{-\ell^2/2} {}^\circ_\circ \Big((\alpha_k, \mp\beta)^2 z^{-2} + 2(\alpha_k, \mp\beta) \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \alpha_k(m)z^{-m-2} \\ &\qquad + \sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} \alpha_k(j)\alpha_k(m)z^{-j-m-2} \Big) \sum_{n\in\mathbb{Z}} E^{\mp}(n)z^{-n} {}^\circ_\circ u \otimes e^{\pm\beta}. \end{aligned}$$

We wish to calculate the trace of $o(v(\lambda)) = \mathrm{Res}_z\, z^{\ell^2/2+1}Y(v(\lambda), z)$. By equating the coefficients of $z^{-2-\ell^2/2}$ in the above equations, we have

$$\begin{aligned} o\big(\pm\alpha_1(-2) \otimes e^{\pm\lambda}\big)u \otimes e^{\mp\beta} = \Big(&\frac{\ell}{2}E^{\mp}(0)u \\ &\pm \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} (-m-1) {}^\circ_\circ \alpha_1(m)E^{\mp}(-m) {}^\circ_\circ u\Big) \otimes e^{\pm\beta}, \end{aligned} \tag{8}$$

$$o(\alpha_i(-1)^2 \otimes e^{\pm\lambda})u \otimes e^{\mp\beta} = \Big(\sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^\circ_\circ \alpha_i(j)\alpha_i(m) {}^\circ_\circ E^{\mp}(-j-m)u\Big) \otimes e^{\pm\beta}, \tag{9}$$

$$o(\alpha_1(-1)^2\otimes e^{\pm\lambda})u\otimes e^{\mp\beta} = \Big(\frac{\ell^2}{4}E^{\mp}(0)u \mp \ell \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} {}^\circ_\circ\, \alpha_1(m)E^{\mp}(-m)\, {}^\circ_\circ\, u$$
$$+\sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^\circ_\circ\, \alpha_1(j)\alpha_1(m)E^{\mp}(-j-m)\, {}^\circ_\circ\, u\Big)\otimes e^{\pm\beta}. \quad (10)$$

Combining equations (8), (9) and (10) as indicated by the definition of v, equation (7), gives

$$o(v\otimes e^{\lambda})u\otimes e^{-\beta} = \big(O^{+}u\big)\otimes e^{\beta}$$
$$o(t(v)\otimes e^{-\lambda})u\otimes e^{\beta} = \big(O^{-}u\big)\otimes e^{-\beta}.$$

where

$$O^{\pm} := -\frac{\ell^2}{4}E^{\mp}(0) \pm \ell \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} m\, {}^\circ_\circ\, \alpha_1(m)E^{\mp}(-m)\, {}^\circ_\circ$$
$$+(2\ell^2-1)\sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^\circ_\circ\, \alpha_i(j)\alpha_i(m)\, {}^\circ_\circ\, E^{\mp}(-j-m) + \sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^\circ_\circ\, \alpha_1(j)\alpha_1(m)E^{\mp}(-j-m)\, {}^\circ_\circ\, .$$

To compute the trace of $o(v(\lambda))q^{L(0)}$ on $V(\beta)^+$ we use the decomposition

$$V(\beta)^+ = M(1)^+\otimes\mathbb{C}\left(e^{\beta}+e^{-\beta}\right)\oplus M(1)^-\otimes\mathbb{C}\left(e^{\beta}-e^{\beta}\right)$$

in conjunction with the following lemma.

Lemma 4.3.

$$\operatorname{tr} o(v(\lambda))q^{L(0)}\Big|_{M(1)^+\otimes\mathbb{C}(e^{\beta}+e^{-\beta})} = q^{\ell^2/8}\operatorname{tr} O^{\pm}q^{L(0)}\Big|_{M(1)^+}$$
$$\operatorname{tr} o(v(\lambda))q^{L(0)}\Big|_{M(1)^-\otimes\mathbb{C}(e^{\beta}-e^{-\beta})} = -q^{\ell^2/8}\operatorname{tr} O^{\pm}q^{L(0)}\Big|_{M(1)^-}$$

Proof. We prove the second equation. The proof of the first is very similar. Let $\{u_\gamma|\gamma\in C\}$ be a basis of $M(1)^-$; $\{u_\gamma\otimes(e^{\beta}-e^{-\beta})|\gamma\in C\}$ is a basis of $M(1)^-\otimes\mathbb{C}(e^{\beta}-e^{-\beta})$. We apply $o(v(\lambda))q^{L(0)}$ to $u_\alpha\otimes(e^{\beta}-e^{-\beta})$, project into $V(\beta)$ and get

$$\Big(-\big(O^{+}u_\alpha\big)\otimes e^{\beta}+\big(O^{-}u_\alpha\big)\otimes e^{-\beta}\Big)q^{\mathrm{wt}\,u_\alpha+\ell^2/2}. \quad (11)$$

Because t preserves $V(\beta)$, and $o(v(\lambda))q^{L(0)}u_\alpha$ and $(e^{\beta}-e^{-\beta})$ are in $V_\Lambda^+[q]$, (11) is in $V(\beta)^+[q]$, in particular it is fixed by t. Thus

$$-t\big(O^{+}u_\alpha\big) = O^{-}u_\alpha. \quad (12)$$

Let $\{v_\delta | \delta \in D\}$ be a basis of $M(1)^+$, and write $O^+ u_\alpha = \sum_{\gamma\in C} c_\gamma u_\gamma + \sum_{\delta\in D} c_\delta v_\delta$ and $O^- u_\alpha = \sum_{\gamma\in C} d_\gamma u_\gamma + \sum_{\delta\in D} d_\delta v_\delta$ for some $c_\gamma, c_\delta, d_\gamma, d_\delta \in \mathbb{C}$. From (12), $c_\gamma = d_\gamma$ and $-c_\delta = d_\delta$. We rewrite (11) as

$$q^{\ell^2/8}\Big(- \sum_{\gamma\in C} c_\gamma u_\gamma \otimes \left(e^\beta - e^{-\beta}\right) - \sum_{\delta\in D} c_\delta v_\delta \otimes \left(e^\beta + e^{-\beta}\right)\Big) q^{\mathrm{wt}\, u_\alpha}.$$

The projection into the span of $u_\alpha \otimes (e^\beta - e^{-\beta})$ is $-q^{\ell^2/8} c_\alpha q^{\mathrm{wt}\, u_\alpha}$; thus

$$\mathrm{tr}\, o(v(\lambda)) q^{L(0)}\Big|_{M(1)^-\otimes\mathbb{C}(e^\beta - e^{-\beta})} = -q^{\ell^2/8} \sum_{\gamma\in C} c_\gamma q^{\mathrm{wt}\, u_\gamma}.$$

Clearly $\mathrm{tr}\, O^\pm q^{L(0)}|_{M(1)^-} = \sum_{\gamma\in C} c_\gamma q^{\mathrm{wt}\, u_\gamma}$ and the result follows. □

Hence it remains to compute the trace of $O^\pm q^{L(0)}$ on $M(1)^\pm$. To do this we define $x^N \in (\mathrm{End}\, M(1))\,[x]$ by

$$x^N \left(\alpha_1(-n_1)\alpha_2(-n_2)\ldots\alpha_k(-n_k)\right) = x^k \alpha_1(-n_1)\alpha_2(-n_2)\ldots\alpha_k(-n_k) \qquad (13)$$

for $\alpha_i \in \mathfrak{h}$ and $n_i > 0$ and compute the trace of $O^\pm x^n q^{L(0)}$ on $M(1)$.

Lemma 4.4.

$$\mathrm{tr}\, O^\pm x^N q^{L(0)}\Big|_{M(1)} = \frac{\ell^2}{4}\left(-1 + 24\sum_{n>0}\frac{nxq^n}{1-xq^n}\right)\frac{\exp\left(\sum_{n>0}\frac{-\ell^2 xq^n}{n(1-xq^n)}\right)}{\prod_{n>0}(1-xq^n)^{24}}$$

Proof. Dong and Mason [DM00, Lemma 4.2] show that

$$\mathrm{tr}\, E^\pm(0) x^N q^{L(0)}\Big|_{M(1)} = \frac{\exp\left(\sum_{n>0}\frac{-\ell^2 xq^n}{n(1-xq^n)}\right)}{\prod_{n>0}(1-xq^n)^{24}}.$$

We compute the traces of the remaining components of $O^\pm$ using their methodology.

Let $A = \mathbb{C}\alpha_1$, $B = \mathbb{C}\alpha_i$ and $\mathfrak{h} = A \oplus B \oplus C$ be an orthogonal direct sum, then

$$M(1) = S(\hat{\mathfrak{h}}^-) = S(\hat{A}^-) \otimes S(\hat{B}^-) \otimes S(\hat{C}^-).$$

Furthermore, if T, U, V and W are linear endomorphisms such that $T|_{M(1)} = U|_{S(\hat{A}^-)} \otimes V|_{S(\hat{B}^-)} \otimes W|_{S(\hat{C}^-)}$, then

$$trT = \mathrm{tr}\, U \cdot \mathrm{tr}\, V \cdot \mathrm{tr}\, W.$$

It is not hard to see that

$$\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} m \,{}^\circ_\circ\, \alpha_1(m)E^{\mp}(-m)\,{}^\circ_\circ\, x^N q^{L(0)}\Big|_{M(1)}$$

$$= \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} m \,{}^\circ_\circ\, \alpha_1(m)E^{\mp}(-m)\,{}^\circ_\circ\, x^N q^{L(0)}\Big|_{S(\hat{A}^-)} \otimes\ x^N q^{L(0)}\Big|_{S((\widehat{B\oplus C})^-)},$$

$$\sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^\circ_\circ\, \alpha_1(j)\alpha_1(m)E^{\mp}(-j-m)\,{}^\circ_\circ\, x^N q^{L(0)}\Big|_{M(1)}$$

$$= \sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^\circ_\circ\, \alpha_1(j)\alpha_1(m)E^{\mp}(-j-m)\,{}^\circ_\circ\, x^N q^{L(0)}\Big|_{S(\hat{A}^-)} \otimes\ x^N q^{L(0)}\Big|_{S((\widehat{B\oplus C})^-)}.$$

For $\sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^\circ_\circ\, \alpha_i(j)\alpha_i(m)\,{}^\circ_\circ\, E^{\mp}(-j-m)$, write a generic element of $M(1)$ as pq, where $p \in S(\hat{B}^-)$ and $q \in S((\widehat{A\oplus C})^-)$. Here p is a polynomial in $\alpha_i(-n)$'s $n \in \mathbb{N}$, of degree k. Assume that ${}^\circ_\circ\, \alpha_i(m)\alpha_i(j)\,{}^\circ_\circ\, E^{\mp}(-m-j)$ has a non-zero contribution to the trace on $M(1)$. It must preserve the degree of p. The terms of $E^{\mp}(-m-j)$ have no effect on the degree of p, because $(\lambda, \alpha_i) = 0$; however, $\alpha_i(m)\alpha_i(j)$ takes p to 0 or a polynomial of degree $k-(m+j)$. Thus $m+j=0$, and the only terms of $\sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^\circ_\circ\, \alpha_i(j)\alpha_i(m)\,{}^\circ_\circ\, E^{\mp}(-j-m)pq$ that contribute to the trace are $\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} {}^\circ_\circ\, \alpha_i(m)\alpha_i(-m)\,{}^\circ_\circ\, E^{\mp}(0)pq$. Furthermore

$$\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} {}^\circ_\circ\, \alpha_i(m)\alpha_i(-m)\,{}^\circ_\circ\, E^{\mp}(0)x^N q^{L(0)}\Big|_{M(1)} = E^{\mp}(0)x^N q^{L(0)}\Big|_{S(\hat{A}^-)}$$

$$\otimes\ 2\sum_{m\in\mathbb{N}} \alpha_i(-m)\alpha_i(m)x^N q^{L(0)}\Big|_{S(\hat{B}^-)} \otimes\ x^N q^{L(0)}\Big|_{S(\hat{C}^-)}.$$

We know that for any vector space W of dimension k

$$\operatorname{tr} x^N q^{L(0)}\Big|_{S(\hat{W}^-)} = \prod_{n>0} \frac{1}{(1-xq^n)^k}.$$

Additionally, Dong and Mason [DM00, Lemmma 4.1] show that

$$\operatorname{tr} E^{\pm}(0)x^N q^{L(0)}\Big|_{S(\hat{A}^-)} = \frac{\exp\left(\sum_{n>0} \frac{-\ell^2 xq^n}{n(1-xq^n)}\right)}{\prod_{n>0}(1-xq^n)}.$$

The remaining traces necessary to find $\operatorname{tr} O^{\pm}x^N q^{L(0)}\big|_{M(1)}$ are given in the following lemma.

Lemma 4.5. *We have*

$$\operatorname{tr}\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} m \,{}^{\circ}_{\circ}\, \alpha_1(m)E^{\mp}(-m)\,{}^{\circ}_{\circ}\, x^N q^{L(0)}\Big|_{S(\hat{A}^-)} =$$

$$\pm 2\ell \sum_{n>0} \frac{nxq^n}{1-xq^n} \frac{\exp\left(\sum_{n>0}\frac{-\ell^2 xq^n}{n(1-xq^n)}\right)}{\prod_{n>0} 1-xq^n}, \quad (14)$$

$$\operatorname{tr}\sum_{m\in\mathbb{N}} \alpha_i(-m)\alpha_i(m)x^N q^{L(0)}\Big|_{S(\hat{B}^-)} = \sum_{n>0}\frac{nxq^n}{1-xq^n}\prod_{n>0}\frac{1}{1-xq^n}, \quad (15)$$

$$\operatorname{tr}\sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^{\circ}_{\circ}\, \alpha_1(j)\alpha_1(m)E^{\mp}(-j-m)\,{}^{\circ}_{\circ}\, x^N q^{L(0)}\Big|_{S(\hat{A}^-)} =$$

$$2\sum_{n>0} \frac{nxq^n}{1-xq^n} \frac{\exp\left(\sum_{n>0}\frac{-\ell^2 xq^n}{n(1-xq^n)}\right)}{\prod_{n>0} 1-xq^n}. \quad (16)$$

Proof. We prove equation (14) in some detail and sketch proofs of equations (15) and (16)

The set $\{\alpha_1(-n)^{k_n}\alpha_1(-(n-1))^{k_{n-1}}\cdots\alpha_1(-1)^{k_1} | k_i \geq 0, n \geq 1\}$ is a basis for $S(\hat{A}^-)$. Fix a basis element $\boldsymbol{\alpha} = \alpha_1(-n)^{k_n}\alpha_1(-(n-1))^{k_{n-1}}\cdots\alpha_1(-1)^{k_1}$, and for convenience set $m\,{}^{\circ}_{\circ}\,\alpha_1(m)E^{\mp}(-m)\,{}^{\circ}_{\circ} = f^{\mp}(m)$. We find the projection, $P_{\boldsymbol{\alpha}}$ into the space $\mathbb{C}\boldsymbol{\alpha}$ of $\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} f^{\mp}(m)\boldsymbol{\alpha}$. Only those summands that preserve the degree of all of the $\alpha_1(-j)$ in $\boldsymbol{\alpha}$ will have non-zero contributions to this projection. For $\sum_{m>0} f^{\mp}(m)$ these are

$$\sum_{m=1}^{n}\sum_{\substack{p_j=0\\ 0\leq j\leq n\\ j\neq m}}^{k_j}\sum_{p_m=0}^{k_m-1} m\frac{\big(\pm\lambda(-1)\big)^{p_1}}{p_1!}\cdots\frac{\big(\pm\lambda(-m)\big)^{p_m+1}}{m^{p_m+1}(p_m+1)!}\cdots$$
$$\frac{\big(\pm\lambda(-n)\big)^{p_n}}{n^{p_n}p_n!}\frac{\big(\mp\lambda(1)\big)^{p_1}}{p_1!}\cdots\frac{\alpha_1(m)\big(\mp\lambda(m)\big)^{p_m}}{m^{p_m}p_m!}\cdots\frac{\big(\mp\lambda(n)\big)^{p_n}}{n^{p_n}p_n!}. \quad (17)$$

Using equation (17) and the actions of the $\alpha_1(n)$'s given in equation (3), we calculate:

$$P_{\boldsymbol{\alpha}}\Big(\sum_{m>0} f^{\mp}(m)\boldsymbol{\alpha}\Big) = \pm \sum_{m=1}^{n} \sum_{\substack{p_j=0\\1\le j\le n}}^{k_j} (-1)^{p_1+\cdots+p_m} m$$

$$\cdot \frac{\ell^{2p_m+1} m^{p_m+1} k_m \cdots (k_m-p_m)}{m^{2p_m+1}(p_m+1)!p_m!} \prod_{\substack{j=1\\j\neq m}}^{n} \frac{\ell^{2p_j} j^{p_j} k_j \cdots (k_j-p_j+1)}{j^{2p_j}(p_j!)^2}\boldsymbol{\alpha},$$

$$= \pm\ell \sum_{m=1}^{n} \sum_{\substack{p_j=0\\1\le j\le n}}^{k_j} m\frac{\binom{k_m}{p_m+1}}{p_m!}\left(\frac{-\ell^2}{m}\right)^{p_m} \prod_{\substack{j=1\\j\neq m}}^{n} \frac{\binom{k_j}{p_j}}{p_j!}\left(\frac{-\ell^2}{j}\right)^{p_j}\boldsymbol{\alpha}.$$

A similar calculation for $m > 0$ shows that $P_{\boldsymbol{\alpha}}(\sum_{m<0} f^{\mp}(m)\boldsymbol{\alpha})$ is the same as $P_{\boldsymbol{\alpha}}(\sum_{m>0} f^{\mp}(m)\boldsymbol{\alpha})$. So $P_{\boldsymbol{\alpha}}(\sum_{\substack{m\in\mathbb{Z}\\m\neq 0}} f^{\mp}(m)\boldsymbol{\alpha}) = 2P_{\boldsymbol{\alpha}}(\sum_{m>0} f^{\mp}(m)\boldsymbol{\alpha})$.

To compute the trace we sum the above coefficient of $\boldsymbol{\alpha}$ over all possible choices of $\boldsymbol{\alpha}$, that is over all n's and k_j's. Let $\mathbb{Z}^\infty = \bigoplus_{j\geq 1}\mathbb{Z}$, i.e., the space of $\mathbf{n} = (n_1, n_2, \ldots, n_j, \ldots)$ such that $n_j \in \mathbb{Z}$ for all j and $n_j = 0$ for all but finitely many j. Put a partial ordering on $\mathbb{Z}^\infty$ such that $\mathbf{n} \leq \mathbf{m}$ if $n_j \leq m_j$ for all j. Then

$$\operatorname{tr}\sum_{\substack{m\in\mathbb{Z}\\m\neq 0}} m\,{}^{\circ}_{\circ}\,\alpha_1(m)E^{\mp}(-m)\,{}^{\circ}_{\circ}\,x^N q^{L(0)}\Big|_{S(\hat{A}^-)} = \pm 2\ell \sum_{m>0} m \sum_{\substack{\mathbf{k}\in\mathbb{Z}^\infty\\ \mathbf{k}\geq\mathbf{0}}} \sum_{\substack{\mathbf{p}\in\mathbb{Z}^\infty\\ \mathbf{0}\leq\mathbf{p}\leq\mathbf{k}}} \frac{\binom{k_m}{p_m+1}}{p_m!}$$

$$\cdot\left(\frac{-\ell^2}{m}\right)^{p_m} \prod_{\substack{j\geq 1\\j\neq m}} \frac{\binom{k_j}{p_j}}{p_j!}\left(\frac{-\ell^2}{j}\right)^{p_j} x^{\sum_{j\geq 0}k_j} q^{\sum_{j\geq 0} jk_j},$$

which equals

$$\pm 2\ell \sum_{m>0} m\Bigg(\sum_{k_m,p_m\geq 0} \frac{\binom{k_m}{p_m+1}}{p_m!}\left(\frac{-\ell^2}{m}\right)^{p_m} x^{k_m} q^{mk_m}\Bigg)$$

$$\cdot \prod_{\substack{k_j,p_j\geq 0\\ j\geq 1\\ j\neq m}} \Bigg(\sum_{k_j,p_j\geq 0} \frac{\binom{k_j}{p_j}}{p_j!}\left(\frac{-\ell^2}{j}\right)^{p_j} x^{k_j} q^{jk_j}\Bigg).$$

We can now rearrange this sum as Dong and Mason do in the proof of their lemma 4.1 [DM00] and get equation (14).

We now turn to equation (15). Fix a basis element $\boldsymbol{\alpha} = \alpha_i(-n)^{k_n}\alpha_i(-(n-1))^{k_{n-1}}\cdots\alpha_i(-1)^k_1$ of $S(\hat{B}^-)$.

$$\sum_{m>0}\alpha_i(-m)\alpha_i(m)\boldsymbol{\alpha} = \sum_{m>0} mk_m\boldsymbol{\alpha}$$

Add over all basis elements to compute the trace;

$$\operatorname{tr} \sum_{m>0} \alpha_i(-m)\alpha_i(m) x^N q^{L(0)} \Big|_{S(\hat{B}^-)} = \sum_{\substack{\mathbf{k} \in \mathbb{Z}^\infty \\ \mathbf{k} \geq \mathbf{0}}} \sum_{m>0} m k_m \prod_{j>0} x^{k_j} q^{j k_j}.$$

We have

$$\sum_{k_m > 0} m k_m x^{k_m} q^{m k_m} = q \frac{\partial}{\partial q} \left(\frac{1}{1 - x q^m} \right) = \frac{m x q^m}{(1 - x q^m)^2}.$$

So summing over the k_j's in $\mathbf{k}$ we conclude that

$$\operatorname{tr} \sum_{m>0} \alpha_i(-m)\alpha_i(m) x^N q^{L(0)} \Big|_{S(\hat{B}^-)} = \sum_{m>0} \frac{m x q^m}{1 - x q^m} \prod_{j>0} \frac{1}{1 - x q^j}.$$

To prove equation (16) write $\sum_{\substack{j,m \in \mathbb{Z} \\ j,m \neq 0}} {}^\circ_\circ \alpha_1(j)\alpha_1(m) E^{\mp}(-j-m) {}^\circ_\circ$ as

$$\sum_{\substack{j,m \in \mathbb{Z} \\ j,m \neq 0 \\ j \neq \pm m}} {}^\circ_\circ \alpha_1(j)\alpha_1(m) E^{\mp}(-j-m) {}^\circ_\circ$$
$$+ \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} {}^\circ_\circ \alpha_1(m)^2 E^{\mp}(-2m) {}^\circ_\circ + \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} {}^\circ_\circ \alpha_1(-m)\alpha_1(m) E^{\mp}(0) {}^\circ_\circ.$$

The $m \neq \pm j$ terms make no contribution to the trace because each pair (m, j) with $j > 0$ is cancelled by $(m, -j)$.

Calculating the second and third sums as in the proof of equation (14), we find:

$$\operatorname{tr} \sum_{\substack{j,m \in \mathbb{Z} \\ j,m \neq 0}} {}^\circ_\circ \alpha_1(j)\alpha_1(m) E^{\mp}(-j-m) {}^\circ_\circ x^N q^{L(0)} \Big|_{S(\hat{A}^-)} =$$
$$2 \sum_{m>0} \frac{x q^m}{(1 - x q^m)^2} \sum_{p_m \geq 0} \left(\ell^2 \frac{x q^m}{1 - x q^m} + m(p_m + 1) \right)$$
$$\cdot \frac{1}{p_m!} \left(\frac{-\ell^2 x q^m}{m(1 - x q^m)} \right)^{p_m} \prod_{\substack{j>0 \\ j \neq m}} \frac{1}{1 - x q^j} \sum_{p_j \geq 0} \frac{1}{p_j!} \left(\frac{-\ell^2 x q^j}{j(1 - x q^j)} \right)^{p_j}$$
$$= \quad 2 \sum_{m>0} \frac{m x q^m}{1 - x q^m} \prod_{j>0} \frac{1}{1 - x q^j} \sum_{p_j > 0} \frac{1}{p_j!} \left(\frac{-\ell^2 x q^n}{n(1 - x q^n)} \right)^{p_j}.$$

The last equality follows from the identity

$$m \sum_{p_m \geq 0} \frac{p_m}{p_m!} \left(\frac{-\ell^2 x q^m}{m(1 - x q^m)} \right)^{p_m} = -\ell^2 \frac{x q^m}{1 - x q^m} \sum_{p_m \geq 0} \frac{1}{p_m!} \left(\frac{-\ell^2 x q^m}{m(1 - x q^m)} \right)^{p_m},$$

and equation (16) now follows easily. □

This also completes the proof of lemma 4.4. $\square$

We introduce the notation

$$f(q,x) := \frac{\exp\left(\sum_{n>0} \frac{-\ell^2 x q^n}{n(1-xq^n)}\right)}{\prod_{n>0}(1-xq^n)^{24}}, \qquad T(q,x) := \sum_{n>0} \frac{nxq^n}{1-xq^n},$$

and rewrite lemma 4.4 in terms of this notation;

$$\operatorname{tr} O^{\pm} x^N q^{L(0)}\Big|_{M(1)} = \frac{\ell^2}{4}\big(-1+24T(q,x)\big)f(q,x).$$

It is not hard to see that

$$\operatorname{tr} O^{\pm} q^{L(0)}\Big|_{M(1)^+} = \frac{\ell^2}{8}\Big(\big(-1+24T(q,1)\big)f(q,1) + \big(-1+24T(q,-1)\big)f(q,-1)\Big), \tag{18}$$

$$\operatorname{tr} O^{\pm} q^{L(0)}\Big|_{M(1)^-} = \frac{\ell^2}{8}\Big(\big(-1+24T(q,1)\big)f(q,1) - \big(-1+24T(q,-1)\big)f(q,-1)\Big). \tag{19}$$

Recall that the trace of $o(v(\lambda))$ on V_Λ^+ equals the trace on $V(\beta)^+$ which equals the sum of the traces on $M(1)^+ \otimes (e^\beta + e^{-\beta})$ and $M(1)^- \otimes (e^\beta - e^{-\beta})$. From lemma 4.3 and equations (18) and (19), we have

$$\operatorname{tr} o(v(\lambda)) q^{L(0)}\Big|_{V_\Lambda^+} = q^{\ell^2/8}\frac{\ell^2}{4}\big(-1+24T(q,-1)\big)f(q,-1) \tag{20}$$

From lemmas 4.3 and 4.4 of Dong and Mason [DM00]

$$q^{\frac{\ell^2}{8}-1} f(q,-1) = \eta(\tau)^{12}\left(\frac{\Theta_1(\tau)}{2}\right)^{\ell^2-12}, \tag{21}$$

where $\eta(\tau) := q^{1/8}\prod_{n>0}(1-q^n)$ is the Dedekind eta function and $\Theta_1(\tau) := 2q^{1/8}\prod_{n>0}(1-q^n)(1+q^n)^2$ is a Jacobi theta function. (For more about $\eta(\tau)$ and $\Theta_i(\tau)$, see [Cha85].)

To rewrite $-1+24T(q,-1)$, we multiply the each term in the sum $T(q,-1) = -\sum_{n>0}\frac{nq^n}{1+q^n}$ by $\frac{1-q^n}{1-q^n}$, add and subtract nq^n from the numerator and split the result into two sums to obtain

$$T(q,-1) = -\sum_{n>0}\frac{nq^n}{1-q^n} + 2\sum_{n>0}\frac{nq^{2n}}{1-q^{2n}}.$$

The function $E_2(\tau) := 1 - 24\sum_{n>0}\sigma_1(n)q^n = 1 - 24\sum_{n>0}\frac{nq^n}{1-q^n}$. So

$$-1+24T(q,-1) = E_2(\tau) - 2E_2(2\tau). \tag{22}$$

The contribution of V_Λ^+ to $Z_{V^\natural}(v,\tau)$ is $q^{-1}\operatorname{tr} o(v)q^{L(0)}\big|_{V_\Lambda^+}$, and equations (20), (21) and (22) yield the following lemma.

Lemma 4.6. *The contribution of V_Λ^+ to $Z_{V^\natural}(v(\lambda),\tau)$ is*

$$\frac{\ell^2}{8}\eta(\tau)^{12}2\big(E_2(\tau)-2E_2(2\tau)\big)\left(\frac{\Theta_1(\tau)}{2}\right)^{\ell^2-12}.$$

Remark 4.7. *The function E_2 has q-expansion analogous to the Eisenstein series. However, it is modular of level two not one. This is to be expected as V_Λ^+ is rational but not holomorphic.*

Recall that $V^\natural = V_\Lambda^+ \oplus (V_\Lambda^T)^+$. We must now compute the contribution of $(V_\Lambda^T)^+$ to the graded trace of $v(\lambda)$. A twisted module for V_Λ, V_Λ^T is the tensor product of two spaces, $S(\mathfrak{h}[-1]^-)$ and T. The first, $S(\mathfrak{h}[-1]^-)$, is quite similar to $M(1)$, described in section 2. It is a module for $\mathfrak{h}[-1] := \mathfrak{h}\otimes t^{\frac{1}{2}}\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c$ induced from the one-dimensional module for the non-negatively graded subalgebra of $\mathfrak{h}[-1]$ on which c acts as the identity and the positively graded subalgebra acts as zero. The second part, T, is the projective 2^{12}-dimensional representation for Λ such that 2Λ acts trivially. The involution t acts on $S(\mathfrak{h}[-1]^-)$ in the same way it does on the $M(1)$ in V_Λ (see equation (5)) and on T as multiplication by -1. Thus the elements of V_Λ^T fixed by t are

$$\left(V_\Lambda^T\right)^+ = S(\mathfrak{h}[-1]^-)^-\otimes T.$$

The action of V_Λ^+ on $\left(V_\Lambda^T\right)^+$ is given by the twisted vertex operator

$$Y_{\mathbb{Z}+\frac{1}{2}}(v,z) := Y_0(\exp(\Delta_z)v,z).$$

Here for $w=\alpha_1(-n_1)\cdots\alpha_k(-n_k)\otimes e^\gamma\in V_\Lambda$

$$\begin{aligned}
Y_0(w,z) &:= {\circ\atop\circ}\frac{1}{(n_1-1)!}\left(\frac{\partial}{\partial z}\right)^{n_1-1}\alpha_{\frac{1}{2},1}(z)\cdots\frac{1}{(n_k-1)!}\left(\frac{\partial}{\partial z}\right)^{n_k-1}\alpha_{\frac{1}{2},k}(z)\\
&\qquad\qquad\cdot 2^{-(\gamma,\gamma)}E^-_{\frac{1}{2}}(-\gamma,z)E^+_{\frac{1}{2}}(-\gamma,z)\gamma z^{\frac{-(\gamma,\gamma)}{2}}{\circ\atop\circ},\\
\alpha_{\frac{1}{2}}(z) &:= \sum_{n\in\mathbb{Z}+\frac{1}{2}}\alpha(n)z^{-n-1},\\
E^{\pm}_{\frac{1}{2}}(\alpha,z) &:= \exp\Bigg(\sum_{n>0}\frac{\alpha\left(\pm(n+\frac{1}{2})\right)}{\pm(n+\frac{1}{2})}z^{\mp(n+\frac{1}{2})}\Bigg).
\end{aligned}$$

Each $\alpha(n)\in\mathfrak{h}[-1]$ acts on $S(\mathfrak{h}[-1]^-)$ and γ acts on T. The linear map $\exp(\Delta_z):V_\Lambda\to V_\Lambda\{z\}$ is defined in Frenkel, Lepowsky and Meurman [FLM88, section 9.2].

The space $(V_\Lambda^T)^+$ is integrally graded by defining the weight of an element $\alpha_1(-n_1)\alpha_2(-n_2)\cdots\alpha_k(-n_k)\otimes t$ to be $\frac{3}{2}+\sum_{i=1}^k n_i$. The contribution of the twisted space to the graded trace of the element $v(\lambda)$ on $V^\natural$ is

$$q^{-1}\sum_{n\geq 1}\operatorname{tr}\big|_{(V_\Lambda^T)^+_n} o_{\frac{1}{2}}(v(\lambda))q^n$$

where $o_{\frac{1}{2}}(v) := \operatorname{Res}_z z^{\mathrm{wt}\,v-1}Y_{\mathbb{Z}+\frac{1}{2}}(v,z)$.

Recall that $v(\lambda) = v\otimes e^\lambda + t(v)\otimes e^{-\lambda}$, where $v = -\ell\alpha_1(-2)+(2\ell^2+1)\alpha_i(-1)^2+\alpha_1(-1)^2$. Clearly,

$$Y_{\mathbb{Z}+\frac{1}{2}} = Y_0(\exp(\Delta_z)v\otimes e^\lambda, z) + Y_0(\exp(\Delta_z)t(v)\otimes e^{-\lambda}, z).$$

With respect to our orthonormal basis of $\mathfrak{h}$, $\{\alpha_1,\alpha_2,\cdots\alpha_{24}\}$, Frenkel, Lepowsky and Meurman's equation for $\exp(\Delta_z)$ [FLM88, 9.2.42] becomes

$$\begin{aligned}\exp(\Delta_z) = 1 - \frac{1}{2}\sum_{k=1}^{24}\alpha_k(1)\alpha_k(0)z^{-1} + \frac{1}{16}\sum_{k=1}^{24}\big(3\alpha_k(2)\alpha_k(0)+\alpha_k(1)^2\big)z^{-2}\\ + \frac{1}{8}\left(\sum_{k=1}^{24}\alpha_k(1)\alpha_k(0)\right)^2 z^{-2}+\cdots,\end{aligned}$$

and we calculate

$$\begin{aligned}\exp(\Delta_z)v\otimes e^\lambda &= \left(v-\ell\alpha_1(-1)z^{-1}+\frac{\ell^2}{8}z^{-2}\right)\otimes e^\lambda\\ \exp(\Delta_z)t(v)\otimes e^{-\lambda} &= \left(t(v)+\ell\alpha_1(-1)z^{-1}+\frac{\ell^2}{8}z^{-2}\right)\otimes e^{-\lambda}.\end{aligned}\tag{23}$$

In analogy to the untwisted case, define $E^{\pm}_{\frac{1}{2}}(m)$ by

$$E^-_{\frac{1}{2}}(\pm\lambda,z)E_{\frac{1}{2}}(\pm\lambda,z) = \sum_{n\in\frac{1}{2}\mathbb{Z}} E^{\pm}_{\frac{1}{2}}(n)z^{-n}.$$

Since $\pm\lambda$ is in 2Λ, it acts trivially on V_Λ^T. Thus using the definitions of $o_{\frac{1}{2}}$ and Y_0 and equation (23) we can show that

$$o_{\frac{1}{2}}(v(\lambda)) = 2^{-\ell^2}\big(O^+_{\frac{1}{2}}+O^-_{\frac{1}{2}}\big),\tag{24}$$

where

$$\begin{aligned}O^{\pm}_{\frac{1}{2}} = \mp\ell\sum_{m\in\mathbb{Z}+\frac{1}{2}}(-m-1)\alpha_1(m)E^{\mp}_{\frac{1}{2}}(-m)+(2\ell^2-1)\sum_{j,m\in\mathbb{Z}+\frac{1}{2}}\alpha_i(j)\alpha_i(m)E^{\mp}_{\frac{1}{2}}(-j-m)\\ +\sum_{j,m\in\mathbb{Z}+\frac{1}{2}}\alpha_1(j)\alpha_1(m)E^{\mp}_{\frac{1}{2}}(-j-m)\mp\ell\sum_{m\in\mathbb{Z}+\frac{1}{2}}\alpha_1(m)E^{\mp}_{\frac{1}{2}}(-m)+\frac{\ell^2}{8}E^{\mp}_{\frac{1}{2}}(0).\end{aligned}$$

Again we introduce the operator $x^N \in \operatorname{End} S\big(\hat{\mathfrak{h}}[-1]^-\big)[x]$ (see equation (13)). Using the same methods used to prove lemma 4.4, we can prove the following lemma.

Lemma 4.8.

$$\operatorname{tr} O^{\pm}_{\frac{1}{2}} x^N q^{L(0)}\Big|_{S(\hat{\mathfrak{h}}[-1]^-)} = \frac{\ell^2}{8}\big(1 + 24S(q,x)\big)g(q,x),$$

where $g(q,x)$ is the function used by Dong and Mason [DM00]

$$g(q,x) := q^{3/2}\frac{\exp\Big(\sum_{n>0}\frac{-\ell^2 x q^{n+1/2}}{(n+\frac{1}{2})(1-xq^{n+1/2})}\Big)}{\prod_{n>0}\big(1-xq^{n+1/2}\big)^{24}},$$

and

$$S(q,x) := \sum_{n>0}(2n+1)\frac{xq^{n+1/2}}{1-xq^{n+1/2}}.$$

The trace of $O^{\pm}_{\frac{1}{2}} q^{L(0)}$ on $S(\hat{\mathfrak{h}}[-1]^-)^-$ is the difference of the trace of $O^{\pm}_{\frac{1}{2}} x^n q^{L(0)}$ on $S(\hat{\mathfrak{h}}[-1]^-)$ evaluated at $x = 1$ and the same trace evaluated at $x = -1$. The operator $o_{\frac{1}{2}}(v(\lambda))q^{L(0)}$ acts as the identity on T, thus it contributes 2^{12}, the dimension of T, to the trace. So from equation (24)

$$\begin{aligned}\operatorname{tr} o_{\frac{1}{2}}(v(\lambda))q^{L(0)}\Big|_{(V_\Lambda^T)^+} = 2^{12-\ell^2}\frac{\ell^2}{8}\Big(&\big(1+24S(q,1)\big)g(q,1)\\ &- \big(1+24S(q,-1)\big)g(q,-1)\Big).\end{aligned}$$

Dong and Mason show that

$$2^{12-\ell^2}q^{-1}g(q,1) = \eta(\tau)^{12}\left(\frac{\Theta_2(\tau)}{2}\right)^{\ell^2-12},$$
$$2^{12-\ell^2}q^{-1}g(q,-1) = \eta(\tau)^{12}\left(\frac{\Theta_3(\tau)}{2}\right)^{\ell^2-12},$$

where $\Theta_2(\tau) := \prod_{n>0}(1-q^n)(1-q^{n-1/2})^2$ and $\Theta_3(\tau) := \prod_{n>0}(1-q^n)(1+q^{n-1/2})^2$.

Recall that $T(q,x) = \sum_{n>0}\frac{nxq^n}{1-xq^n}$. By splitting the summation for $T(q^{1/2},x)$ into two sums according to the parity of the index variable, we see that

$$S(q,x) = T(q^{1/2},x) - 2T(q,x).$$

Use this identity and $E_2(\tau) = 1 - 24\sum_{n>0}\frac{nq^n}{1-q^n}$ to obtain

$$1 + 24S(q,1) = -E_2(\tau/2) + E_2(\tau).$$

Using the same identities and equation (22), we can get

$$1 + 24S(q, -1) = E_2(\tau/2) - 4E_2(\tau) + 4E_2(2\tau).$$

Put the above equations together to obtain the following lemma.

Lemma 4.9. *The contribution of* $\left(V_\Lambda^T\right)^+$ *to* $Z_{V^\natural}(v(\lambda), \tau)$ *is*

$$\frac{\ell^2}{8}\eta(\tau)^{12}\Bigg\{\big(-E_2(\tau/2) + 2E_2(\tau)\big)\left(\frac{\Theta_2(\tau)}{2}\right)^{\ell^2-12} + \big(-E_2(\tau/2) + 4E_2(\tau) - 4E_2(2\tau)\big)\left(\frac{\Theta_3(\tau)}{2}\right)^{\ell^2-12}\Bigg\}.$$

The main theorem now follows from lemmas 4.6 and 4.9.

5 Implications of the Main Theorem

In general, the graded trace on $V^\natural$ of a highest-weight vector of positive weight is a cusp form [DM00, Proposition 2]. The Leech lattice is even and has no elements of square length two, so for $\lambda \in 2\Lambda$, $\ell^2 = (\lambda, \lambda)$ is divisible by eight and at least 16. This means that $\mathrm{wt}(v(\lambda)) = (\lambda, \lambda)/2 + 2$ is congruent to two modulo four and at least ten. There are no non-zero cusp forms of weights 10 or 14, so the main theorem for $(\lambda, \lambda) = 16, 24$ gives the identity;

$$2\big(E_2(\tau) - 2E_2(2\tau)\big)\Theta_1(\tau)^n + \big(-E_2(\tau/2) + 2E_2(\tau)\big)\Theta_2(\tau)^n + \big(-E_2(\tau/2 + 4E_2(\tau) - 4E_2(2\tau)\big)\Theta_3(\theta)^n = 0$$

where $n = 4$ or 12.

For $v(\lambda)$ of weight at least 18, i.e., $(\lambda, \lambda) \geq 32$, it is straightforward to show that $Z_{V^\natural}(v(\lambda), \tau)$ is non-zero by computing the coefficient of q. Combining our results with those of Dong and Mason [DM00, Theorem 3], there are $V^\natural$ highest-weight vectors of weights 12 and $2k$ for $k \geq 8$ with non-zero graded trace. The spaces of cusp forms of weights 12, 16, 18, 20, 22 and 26 are one-dimensional; so, for these weights, all cusp forms are achieved as the graded trace of a highest-weight vector.

One of the chief characteristics of the moonshine module is that it admits an action of the monster simple group. Averaging $v(\lambda)$ over the monster produces a monster-invariant highest-weight vector with the same weight and graded trace as $v(\lambda)$. Harada and Lang [HL98] find a generating function for the dimensions of the spaces of monster-invariant weight-k elements of the moonshine module and find the dimension to be one for $k = 12, 16, 18, 20, 22$. For those weights, the cusp form normalized so that the leading term in the q-expansion is q is unique. Thus we have the existence of a unique monster-invariant highest-weight vector $v \in V^\natural$ with $Z_{V^\natural}(v, \tau)$ equal to the normalized cusp form of weight k.

That portion of the construction of $V^\natural$ used in this paper, i.e., the spaces V_Λ^+ and $(V_\Lambda^T)^+$ and the vertex operators $Y(v, z)$ with $v \in V_\Lambda^+$, can be made from any even self-dual lattices, not just the Leech lattice. While Frenkel, Lepowsky and Meurman [FLM88] use the monster in their construction of the vertex operators associated with v in $(V_\Lambda^T)^+$ and proof that $V^\natural$ is a VOA, Dolan, Goddard and Montague [DGM90] perform an analogous construction without relying on the monster. They show that $V_L^+ \oplus (V_L^T)^+$ is a VOA, called a $\mathbb{Z}_{\nvDash}$-orbifold for all even self-dual lattices. Even self-dual lattices with rank 24, Niemeier lattices, produce $\mathbb{Z}_2$-orbifolds with $c = 24$. The calculations in this paper use the central charge but do not depend on any of the special properties of the Leech lattice. Therefore the main theorem also applies to the $\mathbb{Z}_2$-orbifolds of the other Niemeier lattices.

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