MATH 4280 – Exam 1 – 2/13/08 Name: Grading Key

**Hurley**

To receive full credit, all answers must be explained.

1. (20 points) Categorize the following binary codes as:

   (a) uniquely decodable and instantaneous

   (b) uniquely decodable but not instantaneous

   (c) not uniquely decodable but instantaneous

   (d) neither uniquely decodable nor instantaneous

10 • $\{110, 00, 111, 01, 10\}$ (a)

Since no code word is a prefix of another code word, this code is instantaneous and hence for uniquely decodable too.

10 • $\{110, 11, 100, 00, 10\}$ (b)

This code is not a prefix code as $110 = 11.0$, so it is not instantaneous.

To determine unique decodability, we find $C_\infty$.

$C_1 = \{w \in T^+ \mid v = uw, u \& v \in C\}$

$110 = 11.0$ and $100 = 10.0$

So $C_1 = \{0\}$.

$C_2 = \{w \in T^+ \mid v = uw \quad u \in C \& v \in C_1 \text{ or } u \in C_1 \text{ and } v \in C\}$

$00 = 0.0$

So $C_2 = \{0\}$

$\Rightarrow C_n = \{0\}$ for all $n \geq 3$. So $C_\infty = \bigcup_{n=1}^{\infty} C_n = \{0\}$

$\Rightarrow C \cap C_\infty = \emptyset$ so $C$ is uniquely decodable.

2. (20 points) Let $C$ be a code, and let $w$ be an element of $C \cap C_2$. Show that there always exists a code sequence with more than one decoding.

Let $w \in C \cap C^2$.

$C^2 = \{ w \in T^+ \mid v = uw$ with $v \in C$ and $u \in C_1$ or $v \in C_1$ and $u \in C \}$

Case 1 $v \in C$ and $u \in C_1$ , $v =$

10 since $u \in C^1$ there exist $x_1, x_2 \in C$ such that $x_1 = x_2 u$. We also have $v = uw$ with $v, w \in C$

$x_1 . W = x_2 u . w = x_2 . uw = x_2 . V$.

So $x_1 w = x_2 v$ with $x_1, x_2, v, w \in C$.

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10 Case 2 $v \in C^1$ and $u \in C$

Since $v \in C^1$ there exist $x_1, x_2 \in C$ such that $x_1 = x_2 v$. We also have $v = uw$.

thus $x_1 = x_2 uw$ with $x_1, x_2, u, w \in C$.

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In either case there exist a code sequence that is not uniquely decodable

Key

3. (10 points) Does there exist a uniquely decodable ternary code with word lengths 1, 1, 2, 2, 2, 3, 3? Why or why not?

McMillan's Inequality

There is a uniquely decodable r-ary code C with word-lengths $l_1, \ldots, l_q$ if and only if

$$\sum_{i=1}^{q} \frac{1}{r^{l_i}} \leq 1.$$

Here $r = 3$.

$$\sum_{i=1}^{7} \frac{1}{3^{l_i}} = \frac{1}{3} + \frac{1}{3} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{27} + \frac{1}{27}$$

$$= 1 + \frac{2}{27} > 1$$

So no such code exists

4. (35 points) Let $\mathcal{S}$ be a source with probabilities $p_i = \frac{5}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}$.

(a) Give examples of two binary Huffman codes for $S$ with different code-word lengths $\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5\}$.

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Let $S = \{s_1, s_2, s_3, s_4, s_5\}$ with probabilities $5/8,\ 1/8,\ 1/8,\ 1/16,\ 1/16$

Let $s_6 = s_4 \vee s_5 \quad p_6 = 1/8$

$S' = \{s_1, s_2, s_3, s_6\}$ with probabilities $5/8,\ 1/8,\ 1/8,\ 1/8$

Let $s_7 = s_3 \vee s_6 \quad p_7 = 1/4$

$S'' = \{s_1, s_7, s_2\}$ with probabilities $5/8,\ 1/4,\ 1/8$

Let $s_8 = s_7 \vee s_2 \quad p_8 = 3/8$

$S''' = \{s_1, s_8\}$ with probabilities $5/8,\ 3/8$

Let $s_9 = s_1 \vee s_8 \quad p_9 = 1$

$S^{(4)} = \{s_9\}$ with probability $1$

$s_9 \longrightarrow \varepsilon$
$s_1 \longrightarrow 0$
$s_8 \longrightarrow 1$
$s_7 \longrightarrow 10$
$s_2 \longrightarrow 11$
$s_3 \longrightarrow 100$
$s_6 \longrightarrow 101$
$s_4 \longrightarrow 1010$
$s_5 \longrightarrow 1011$

$C_1 = \{0, 11, 100, 1010, 1011\}$ 10

$\ell_1 = 1,\ \ell_2 = 2,\ \ell_3 = 3$

$\ell_4 = \ell_5 = 4$

or

$S' = \{s_1, s_6, s_2, s_3\}$ with probabilities $5/8,\ 1/8,\ 1/8,\ 1/8$

Let $s_7 = s_2 \vee s_3 \quad p_7 = 1/4$

$S'' = \{s_1, s_7, s_6\}$ with probabilities $5/8,\ 1/4,\ 1/8$

Let $s_8 = s_7 \vee s_6 \quad p_8 = 3/8$

$S''' = \{s_1, s_8\}$ with probabilities $5/8,\ 3/8$

Let $s_9 = s_1 \vee s_8 \quad p_9 = 1$

$s_9 \longrightarrow \varepsilon$
$s_1 \longrightarrow 0$
$s_8 \longrightarrow 1$
$s_7 \longrightarrow 10$
$s_6 \longrightarrow 11$
$s_2 \longrightarrow 100$
$s_3 \longrightarrow 101$
$s_4 \longrightarrow 110$
$s_5 \longrightarrow 111$

$C_2 = \{0, 100, 101, 110, 111\}$ 10

$\ell_1 = 1 \quad \ell_2 = \ell_3 = \ell_4 = \ell_5 = 3$

(b) Find the average code length of an optimal binary code for $\mathcal{S}$, the source with probalilities $p_i = \frac{5}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}$ used on the previous page.

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$$L(C) = P_6 + P_7 + P_8 + P_9$$
$$= 1/8 + 1/4 + 3/8 + 1$$
$$= 1 + \frac{1+2+3}{8}$$
$$= 1\,6/8 = 1\,3/4 = \frac{7}{4} = 1.75.$$

(c) Give an example of a optimal binary code for $\mathcal{S}$, the same source again, that is *not* a Huffman code, and explain your answer. (In this question a code is given by specifing a function from $S$ to $T^+$.)

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use the same set of code words as $C_2$
$\{0, 100, 101, 110, 111\}$ switch $w_3$ & $w_4$

let

$D(s_1) = 0$
$D(s_2) = 100$
$D(s_3) = 110$
$D(s_4) = 101$
$D(s_5) = 111$

since $s_6 = s_4 \vee s_5$
In a binary Huffman code 2 first 2 bits in $w_4$ & $w_5$ must be the same

5. (15 points) Let $\mathcal{S}$ be a source with eight symbols $\{s_1, s_2, \ldots, s_8\}$, and let $\mathcal{C}$ be an optimal binary code for $\mathcal{S}$. Prove that the average length of $\mathcal{C}$ is less than or equal to three, i.e., $L(\mathcal{C}) \le 3$.

~~The~~ Let $B = \{0,1\}$ be the binary alphabet and let $C' = \{w \in B^* \mid |w| = 3\}$, that is, the set of all binary words of length 3.

Since all the code-words in $C'$ have the same length, none can be a prefix of any other, so C′ is instantaneous.

Regardless of the probability distribution for S, $L(C') = 3$:

$$L(C') = \sum_{i=1}^{8} l_i p_i = \sum_{i=1}^{8} 3 p_i = 3 \sum_{i=1}^{8} p_i = 3 \cdot 1 = 3.$$

Since C is an optimal binary code, if D is any instantaneous binary code for S,

$$L(C) \le L(D).$$

Therefore $L(C) \le L(C') = 3$.

$$L(C) \le 3$$

(I underlined the key points.)