Sol'ns | HW due 4/4 · Math 4280 Spring 2008

5.1, 5.2, 5.8 + 3/28 additional problem

grade 5.2 out of 4 pts

5.8 out of 8 pts

additional problem out of 8 pts

5.2

Calculate Pr_E where the channel Γ and input Δ are as in example 4.5 (a BSC with $P = 0.8$ and $p = 0.9$), and Δ is the ideal observer rule.

We calculate $(R_{ij}) = \begin{pmatrix} p & 0 \\ 0 & \bar{p} \end{pmatrix} \begin{pmatrix} P & \bar{P} \\ \bar{P} & P \end{pmatrix}$

$$= \begin{pmatrix} .9 & 0 \\ 0 & .1 \end{pmatrix} \begin{pmatrix} .8 & .2 \\ .2 & .8 \end{pmatrix}$$

$$= \begin{pmatrix} .72 & .18 \\ .02 & .08 \end{pmatrix}$$

Since .72 is the largest entry in the first column $\Delta(0) = 0$, and since .18 is the largest entry in the second column $\Delta(1) = 0$, too. thus

$$Pr_C = .72 + .18 = .9$$

So $Pr_E = 1 - .9 = \boxed{.1}$

(Note that this also follows directly from 3/28 additional problem, OK to use that instead).

Exercise 5.8

Let Γ be the BEC with $P > 0$ and let the input probabilities be $p, \bar{p}$ with $0 < p < 1$. Show how to use the binary repetition code R_n to send information through Γ, so that $Pr_E \to 0$ as $n \to \infty$.

The binary repetition code sends 0 through as $\underbrace{00\cdots0}_{n\text{-times}}$ and $\underbrace{11\cdots1}_{n\text{-times}}$.

Since Γ has channel matrix $M = \begin{pmatrix} P & 0 & \bar{P} \\ 0 & P & \bar{P} \end{pmatrix}$, it is not possible for a 0 to become a 1 or vice-versa. Thus no words with both zeros and ones can be received. Let $B_0 = \{0, ?\}$ and $B_1 = \{1, ?\}$ the image of Γ is $(B_0)^n \cup (B_1)^n$. Moreover $(B_0)^n \cap (B_1)^n = \{??\cdots?\}$.

To define a decision rule

$$\Delta : (B_0)^n \cap (B_1)^n \longrightarrow \{00\cdots0, 11\cdots1\}$$

send all elements of $(B_0)^n$ to $000\cdots0$, and all elements of $(B_1)^n$ except $??\cdots?$ to $111\cdots1$. (The assignment of $??\cdots?$ is arbitrary.)

Exercise 5.8 continued

It remains to show that $Pr_E \longrightarrow 0$ as $n \longrightarrow \infty$.

The only possible error would be

$$11\cdots1 \xrightarrow{p^n} ??\cdots? \xrightarrow{\Delta} 00\cdots0 .$$

So $pr_E = pr(\vec{b} = ??\cdots? \mid \vec{a} = 11\cdots1) = \bar{p}^n$

Since $0 < p \leq 1$, $0 \leq \bar{p} < 1$ so

$$\lim_{n\to\infty} pr_E = \lim_{n\to\infty} \bar{p}^n = 0.$$

MATH 4280 – Hurley – homework problem March 26, 2008

1. Prove that for a binary symetric channel an ideal observer rule has $\Pr_C = \max\{p, \bar{p}, P, \bar{P}\}$.

   Hint: By definition an ideal observer rule maximizes $\Pr_C$ over the set of all decision rules.

Since for the binary symetric channel both input and output alphabets have order 2, there are $2^2 = 4$ decision rules. These are

$$\Delta_1(0)=0,\ \Delta_1(1)=0 \quad \& \quad \Delta_2(0)=0,\ \Delta_2(1)=1 \quad \& \quad \Delta_3(0)=1,\ \Delta_3(1)=0 \quad \& \quad \Delta_4(0)=1,\ \Delta_4(1)=1.$$

The matrix $(R_{ij}) = \begin{pmatrix} p & 0 \\ 0 & \bar{p} \end{pmatrix}\begin{pmatrix} P & \bar{P} \\ \bar{P} & P \end{pmatrix} = \begin{pmatrix} pP & p\bar{P} \\ \bar{p}\bar{P} & \bar{p}P \end{pmatrix}$.

In general $pr_c = \sum_j R_{j^*j}$.

So

$$pr_c(\Delta_1) = pP + p\bar{P} = p$$
$$pr_c(\Delta_2) = pP + \bar{p}P = P$$
$$pr_c(\Delta_3) = \bar{p}\bar{P} + p\bar{P} = \bar{P}$$
$$pr_c(\Delta_4) = \bar{p}\bar{P} + \bar{p}P = \bar{p}$$

In general, $pr_c(\text{ideal observer rule}) = \max_\Delta \{pr_c(\Delta)\}$ where Δ varies over all decision rules.

So in this case

1 $pr_c(\text{ideal observer rule}) = \max\{pr_c(\Delta_1), pr_c(\Delta_2), pr(\Delta_3), pr(\Delta_4)\}$
$= \max\{p, P, \bar{P}, \bar{p}\}$