Math 4280 HW #12 4/18/08

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Additional Problem #1

Let (Γ, A, B) be an information channel with capacity C, and suppose that $H(A|B) = 0$. Prove that for all codes $\mathcal{C} \subseteq A^n$ with rate R, $R \leq C$.

If $H(A|B) = 0$, then $I(A,B) = H(A) - H(A|B) = H(A)$.

From Theorem 3.10 (pg 41) $H(A) \leq \log r$, where $|A| = r$. with equality iff $p_i = \frac{1}{r}$ for all i.

Since the capacity is the maximum value of $I(A,B)$ taken over all probabilities p_i on A, $C = \log r$.

Now for any code $\mathcal{C}$, $\mathcal{C} \subseteq A^n$ thus $|\mathcal{C}| \leq |A^n| = r^n$

so the rate $R = \frac{\log|\mathcal{C}|}{n} \leq \frac{\log r^n}{n} = \log r$

So $R \leq \log r = C$, and $R \leq C$.

Additional Problem #2

For all the channels the input and output alphabets are $F_2^3 = \{000, 001, 010, 100, 101, 110, 111\}$, which has eight elements. Since the input distribution is assumed to be uniform $p_i = \frac{1}{8}$ for all i.

If the input & output alphabets have the same indexing, then the decision rule is assumed to be $\Delta(b_j) = a_j$, that is $j^* = j$

In general $Pr_C = \sum_{j=1}^{8} R_{j^*j}$. here $R_{j^*j} = R_{jj} = p_j P_{jj} = \frac{1}{8} P_{jj}$

So $Pr_C = \frac{1}{8} \sum_{j=1}^{8} P_{jj} = \frac{1}{8} \sum_{i=1}^{8} P_{ii}$

Additional #2
continued

So for each rule,
we need.

$P_{ji} = Pr\{\vec{b} = a_1a_2a_3 \mid \vec{a} = a_1a_2a_3\}$, i.e, the probability that the output equals the input, for all $1 \le j \le 8$.

Denote the probability of correct transmission by the BSC, Γ, as P and the prob of error as $\bar{P}$.

Rule (a)

For the first bit $pr(b = a_i \mid a = a_i) = P$ for $a_i = 0$ or 1.
Whereas for the second & third bits,

$$pr(b = a_i \mid a = a_i) = \tfrac{1}{2} \leftarrow \text{there are 2 equally likely choices.}$$

So $P_{ii} = pr(\vec{b} = a_1a_2a_3 \mid \vec{a} = a_1a_2a_3) = P \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{P}{4}$

$$Pr_c = \frac{1}{8}\sum_{i=1}^{8}\frac{P}{4} = \frac{1}{8}\cdot 8 \cdot \frac{P}{4} = \frac{P}{4}$$

and the error probability is $1 - \frac{P}{4}$.

Rule (b)

Since the only received words are 000 and 111,

$pr(\vec{b} = a_1a_2a_3 \mid \vec{a} = a_1a_2a_3) = 0$ except

for $\vec{a} = 000$ and $\vec{a} = 111$.

For $\vec{a} = 000$, the majority bit is 0, so
$b_1 = 0$ with prob P and $b_2 = b_3 = 0$ automatically
so $pr(\vec{b} = 000 \mid \vec{a} = 000) = P$

Likewise if $\vec{a} = 111$, 1 is sent and the prob that $b_1 = 1$ is P, and $b_2 = b_3 = 1$ automatically

so $pr(\vec{b} = 111 \mid \vec{a} = 000) = P$ also.

Thus $pr_c = \frac{1}{8}\sum_{i=1}^{\infty} P_{ii} = \frac{1}{8}(P + P) = \frac{P}{4}$ and $\boxed{pr_E = 1 - \frac{P}{4}}$

Additional #2

continued.

In case ⓒ the probabilities,

$$P_{ii} = pr\{\vec{b} = a_1 a_2 a_3 \mid \vec{a} = a_1 a_2 a_3\} = P \cdot P \cdot P = P^3,$$

since this is 3 correct transmissions.

So $\quad Pr_c = \frac{1}{8} \sum_{i=1}^{8} P_{ii} = \frac{1}{8} P^3$

and $\quad Pr_E = \boxed{1 - \frac{P^3}{8}}$