MATH 4280 – Hurley – homework problem February 6, 2008

1. Let $S = \{s_1, s_2\}$ have probabilities $p_1 \geq p_2$, and let C^2 be an optimal binary encoding of $S^2 = \{s_1s_1, s_1s_2, s_2s_1, s_2s_2\}$.

   (a) Find a constant k such that $L(C^2) = 2$ if and only if $p_1 \leq k$. Prove your answer.

   (b) Find a formula for $L(C^2)$ when $p_1 > k$.

From problem 2.4, if S is a source with probabilities $p_1 \geq p_2 \geq p_3 \geq p_4$, and C is an optimal code for S, then

$$(*) \qquad L(C) = \begin{cases} 2 & \text{if } p_1 \leq p_3 + p_4 \\ 3 - 2p_1 - p_2 & \text{if } p_1 > p_3 + p_4 \end{cases}$$

Apply this result for S^2 with probability distribution

$p_1^2 \geq p_1p_2 \geq p_1p_2 \geq p_2^2$

$L(C) = 2$ iff $p_1^2 \leq p_1p_2 + p_2^2$.

Since $p_2 = 1 - p_1$, $p_1^2 \leq p_1(1-p_1) + (1-p_1)^2$

$$p_1^2 \leq -p_1^2 + p_1 + 1 - 2p_1 + p_1^2.$$

$$p_1^2 + p_1 - 1 \leq 0.$$

Using the quadratic formula, the roots are

$p_1 = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$, let $\phi = \frac{-1+\sqrt{5}}{2} \sim .618\ldots$

and let $\bar{\phi} = \frac{-1-\sqrt{5}}{2} \sim 1.618\ldots$

then $p_1^2 + p_1 - 1 \leq 0$ for $\bar{\phi} \leq p_1 \leq \phi$, and because

$p_1 \geq 0$ by def'n.

$$L(C) = 2 \text{ iff } p_1 \leq \phi = \frac{-1+\sqrt{5}}{2}.$$

If $p_1 > \gamma$, then the second condition in $(*)$ applies

and $L(C) = 3 - 2p_1^2 - p_1p_2 = 3 - 2p_1^2 - p_1(1-p_1) = 3 - p_1^2 - p_1$