Math 4280

Prove that a code is uniquely decodable if and only if its reverse code is uniquely decodable.

Let $w = x_1 x_2 \cdots x_k$ be a word in a code alphabet T and denote the reverse of w by $w^R = x_k x_{k-1} \cdots x_1$. Observe that if $w_i = x_{i_1} x_{i_2} \cdots x_{i_n}$ and $w_j = x_{j_1} x_{j_2} \cdots x_{j_m}$, then

$$(w_i w_j)^R = (x_{i_1} x_{i_2} \cdots x_{i_n} x_{j_1} x_{j_2} \cdots x_{j_m})^R$$
$$= x_{j_m} x_{j_{m-1}} \cdots x_{j_1} x_{i_n} \cdots x_{i_1}$$
$$= w_j^R w_i^R.$$

Now suppose that $C = \{w_1, w_2, \ldots, w_q\}$ is a code and $C^R = \{w_1^R, w_2^R, \ldots, w_q^R\}$ is its reverse code.
Assume that C^R is uniquely decodable and

$u_1 u_2 \cdots u_m = v_1 v_2 \cdots v_n$ with u_i and v_j in C.

then $(u_1 u_2 \cdots u_m)^R = (v_n v_2 \cdots v_1)^R$ so

$u_m^R u_2^R \cdots u_1^R = v_n^R v_2^R \cdots v_1^R$ with u_i^R and v_j^R in C^R.

Since C^R is uniquely decodable $m = n$ and $u_i^R = v_i^R$ for all $1 \le i \le n$,
And since $(u_i^R)^R = u_i$, $u_i = v_i$.
So C is uniquely decodable.

Conversely assume C is uniquely decodable. Since $(C^R)^R = C$. $(C^R)^R$ is uniquely decodable & by the above argument C^R is uniquely decodable too.