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STRONGLY HOLOMORPHIC c = 24 VERTEX OPERATOR ALGEBRAS AND MODULAR FORMS

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Contents

List of Tables

Katherine Lambert Hurley

Strongly holomorphic $c = 24$ vertex operator algebras and modular forms

Abstract

We study the graded traces (one-point correlation functions) of certain strongly holomorphic vertex operator algebras (VOAs) with central charge $c = 24$. Zhu proved that the graded traces are modular forms [Zhu96]. For each VOA, V, we want to know which modular forms are the graded trace of some (highest-weight) vector in V. In Part 1, we use the finite-dimensional reductive Lie algebra V_1 in V and its associated affine Lie algebra in $\operatorname{End} V$ to show that: If V_1 is semisimple, then for any holomorphic modular form f, there exists an element of the VOA with graded trace equal to f. In Part 2, we define a family of highest-weight vectors in the moonshine module. Imitating calculations of Dong and Mason [DM00], we explicitly compute their graded traces and show them to be non-zero cusp forms of weight congruent to two modulo four.

Dedicated to
my mother and my husband,
in repayment.

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Part I

VOAs with V_1 Semisimple

1 Introduction

Physicists devised vertex operator algebras, often called chiral algebras, as key parts of the algebraic formulation of conformal field theories [BPZ84]. The algebraic axioms for vertex operator algebras (VOAs) were introduced in the 1980s by Borcherds and refined by Frenkel, Lepowsky and Meurman [Bor86], [FLM88]. One of the remarkable properties of VOAs is their connection to modular forms. Indeed, the main mathematical motivation for their development was the monstrous moonshine conjectures, which sought to connect the monster simple group with certain modular functions [CN79].

In this work we will deal with strongly holomorphic VOAs, V, with central charge $c = 24$. Holomorphic essentially means that V is the only irreducible V-module, and the central charge is an important invariant of a VOA, related to its structure as a Virasoro module. For the exact definitions of these terms, including the axioms for a vertex operator algebra, please see Section 2.

Assume that V is a strongly holomorphic VOA with central charge 24. Such VOAs have a $\mathbb{Z}$-grading, $V = \oplus_{n\geq 0} V_n$, and for each VOA element, v, there exists $o(v) \in \operatorname{End} V$, which preserves the grading. The graded trace of v is

$$Z(v,q) = q^{-1} \sum_{n\geq 0} \operatorname{tr}\Big|_{V_n} o(v) q^n.$$

Zhu proved that this sum converges to a meromorphic function for $|q| < 1$, holomorphic for $q \neq 0$ [Zhu96]. He also proved that for certain v the graded trace is a meromorphic modular form. To state which v, we need to use an alternate grading on V, $V = \oplus_{n\geq 0} V_{[n]}$, called the square-bracket grading. If $v \in V_{[k]}$, then $Z(v,q)$ is a meromorphic modular form of weight k. If, additionally, $\operatorname{tr}\big|_{V_0} o(v) = 0$, then $Z(v,q)$ is a holomorphic modular from of weight k, which we will refer to simply as a modular form. This leads to the following question, which is the central motivation of this work: For a modular form f, does there exist a (highest-weight) vector v such that $Z(v,q) = f$?

Dong and Mason addressed this question in the specific case of the moonshine module, and found that for any cusp form f there exists an element with graded trace f [DM00]. In a paper with Nagatomo, they determined the space of graded traces of the free boson VOA [DMN01]. In this work, we address the question in a more generic situation.

The smallest possible weight for a non-zero modular form is four. The space of weight-four modular forms is one-dimensional and generated by the Eisenstein series $E_4 = 1/720 + q/3 + \cdots$. So the first non-trivial case of the above question is $f = E_4$. If there exists a v with graded trace E_4, then there exists such a v in $V_{[4]}$, and $\operatorname{tr}\big|_{V_0} o(v) = 0$ and $\operatorname{tr}\big|_{V_1} o(v) \neq 0$. Conversely, if v is in $V_{[4]}$, $\operatorname{tr}\big|_{V_0} o(v) = 0$ and $\operatorname{tr}\big|_{V_1} o(v) \neq 0$, then the graded trace of v is cE_4 for some $c \neq 0$. Most of the effort in the first part of this thesis is directed at finding such v.

In order to work with the elements of $V_{[4]}$, we use an affine Lie algebra structure inside the group of endomorphisms of V. The weight-one homogeneous space, V_1, is a finite-dimensional Lie algebra. In fact, Dong and Mason have shown that in our case it is either semi-simple, abelian or zero [DMa]. The vertex operators of V_1 generate a copy of the untwisted affine Lie algebra $A(V_1)$ in $\operatorname{End}(V)$. In Section 5, we let the endomorphisms in $A(V_1)$ act repeatedly on V_1 to generate elements of V. We then compute the traces on V_0 and V_1 of the most general elements of $V_{[4]}$ that we can obtain this way. In Sections 6 and 8, we use the theory of finite-dimensional semisimple Lie algebras on V_1 to show that particular linear combinations of the general elements have $\operatorname{tr}\big|_{V_0} o(v) = 0$ and $\operatorname{tr}\big|_{V_1} o(v) \neq 0$. Between the two cases we have the existence of such a $v \in V_{[4]}$ for all V with V_1 semisimple. This implies the existence of a vector with graded trace $E_4(\tau)$.

Once we have v such that $Z(v,q) = E_4$, Proposition 3.3 and Proposition 3.2, due to

Dong and Mason, imply that for any modular form f there exists v such that $Z(v,\tau) = f$. This is the main theorem of Part 1. Note that while Proposition 3.3 implies the existence of a highest-weight vector v such that $Z(v,\tau) = E_4$, for the other modular forms f, the v that we show exists, such that $Z(v,\tau) = E_4$, is not a highest-weight vector.

While Part 1 addresses the V_1 semi-simple case, Part 2 concerns the $V_1 = 0$ case. It is a well-known conjecture that the only strongly holomorphic $c = 24$ VOA with $V_1 = 0$ is the moonshine module $V^\natural$. Because $V_1 = 0$, $\mathrm{tr}\big|_{V_1} o(v) = 0$ for all $v \in V^\natural$. Thus if $Z(v,q)$ is holomorphic, it must be a cusp form. The result analogous to that of Part 1 is that for any cusp form f there exists $v \in V^\natural$ such that $Z(v,q) = f$. This was proven by Dong and Mason [DM00]. They also showed that for $k \equiv 0 \mod 12$ and $k \geq 12$ there exists a highest-weight vector such that the graded trace is a non-zero cusp form of weight k. In the second part of this thesis we get the same result for $k \equiv 2 \mod 4$ and $k \geq 18$. This is accomplished through the explicit computation of the graded traces of a family of highest-weight vectors.

2 Definitions and Preliminaries

2.1 Vertex operator algebras

A vertex-operator algebra, $(V, Y, \mathbf{1}, \omega)$ is a complex vector space V, with a linear map $Y : V \mapsto \mathrm{End}\, V[\![z^{-1}, z]\!]$. The image of $v \in V$ under Y is called a *vertex operator* and is written

$$Y(v,z) = \sum_{n\in\mathbb{Z}} v(n) z^{-n-1},$$

where each $v(n)$ is a linear operator on V. The elements $\mathbf{1}$ and ω are called the *vacuum vector* and the *conformal vector*, respectively. The vertex operator of ω defines operators $L(n)$ on V via

$$Y(\omega,z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}.$$

Note that $L(n) = \omega(n+1)$. A vertex-operator algebra, or VOA, must satisfy the following axioms.

1. There exists an integral grading, $V = \bigoplus_{n \geq n_0} V_n$, such that each V_n is a finite-dimensional vector space.
2. For all $u, v \in V$, there exists a constant, N, depending on u and v, such that $u(n)v = 0$ for all $n \geq N$.
3. The vertex operator of the vacuum is the identity; that is, $Y(\mathbf{1}, z) = \mathrm{id}$; alternately, $\mathbf{1}(-1) = \mathrm{id}$ and $\mathbf{1}(n) = 0$ for all other n.
4. The *creation axiom*: $Y(u,z)\mathbf{1} \in V[\![z]\!]$ and $\lim\limits_{z \to 0} Y(u,z)\mathbf{1} = u$; alternately, $u(-1)\mathbf{1} = u$ and $u(n)\mathbf{1} = 0$ for all $n \geq 0$.
5. The *Jacobi identity*:

$$z_0^{-1}\delta\Big(\frac{z_1 - z_2}{z_0}\Big)Y(u,z_1)Y(v,z_2) - z_0^{-1}\delta\Big(\frac{z_2 - z_1}{-z_0}\Big)Y(v,z_2)Y(u,z_1) \\ = z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big)Y(Y(u,z_0)v, z_2). \quad (1)$$

6. The $L(n)$ satisfy the Virasoro Lie algebra commutation relations, i.e.,

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m+n,0}\,\mathrm{id}\,c. \quad (2)$$

   Here $c \in \mathbb{C}$ is a constant called the *central charge* of the VOA.

7. $V_n = \{v \in V | L(0)v = nv\}$
8. For all $v \in V$, $Y(L(-1)v, z) = \frac{d}{dz}Y(v,z)$.

The δ-functions used in the Jacobi identity are formal power series defined as $\delta(z) := \sum_{n \in \mathbb{Z}} z^n$. A term such as $(z_1 - z_0)^n$ is expanded using non-negative powers of the second variable, i.e.,

$$(z_1 - z_0)^n = \sum_{i \geq 0} (-1)^i \binom{n}{i} z_1^{n-i} z_0^i,$$

where $\binom{n}{i} := \frac{n(n-1)\cdots(n-i+1)}{i!}$. By equating the coefficients of $z_0^{-r-1}z_1^{-p-1}z_2^{-q-1}$ in the Jacobi identity and using the identity $(-1)^i\binom{-p-1+i}{i} = \binom{p}{i}$, we get the *Borcherds identities*: For all integers p, q and r,

$$\sum_{i\geq 0}\binom{p}{i}(u(r+i)v)(p+q-i) = \sum_{i\geq 0}(-1)^i\binom{r}{i}\big(u(p+r-i)v(q+i)-(-1)^r v(q+r-i)u(p+i)\big).$$

Set $p=0$ to get the *associativity* relation,

$$(u(r)v)(q) = \sum_{i\geq 0}(-1)^i\binom{r}{i}\big(u(r-i)v(q+i)-(-1)^r v(r+q-i)u(i)\big). \tag{3}$$

Set $r=0$ to get the *commutation* relation,

$$[u(p),v(q)] = \sum_{i\geq 0}\binom{p}{i}(u(i)v)(p+q-i). \tag{4}$$

An element of the subspace V_n is called *homogeneous*. Its *weight* is n, denoted $\operatorname{wt} v = n$. We use part (7) of the definition to compute the weight of $u(n)v$, for homogeneous u and v, with $\operatorname{wt} u = k$ and $\operatorname{wt} v = \ell$.

$$\begin{aligned}L(0)(v(n)u) &= [L(0),v(n)]u + v(n)L(0)u\\ &= [\omega(1),v(n)]u + kv(n)u\end{aligned}$$

Using the commutation relation, equation (4),

$$\begin{aligned}[\omega(1),v(n)] &= (\omega(0)v)(n+1) + (\omega(1)v)(n),\\ &= (L(-1)v)(n+1) + (L(0)v)(n),\\ &= (L(-1)v)(n+1) + \ell v(n).\end{aligned}$$

Equating the coefficients of z^{-n-2} in $Y(L(-1)v,z) = \frac{d}{dz}Y(v,z)$ gives

$$(L(-1)v)(n+1) = (-n-1)v(n). \tag{5}$$

Thus $L(0)(v(n)u) = (k+\ell-n-1)v(n)u$. So

$$\operatorname{wt} v(n)u = \operatorname{wt} v + \operatorname{wt} u - n - 1. \tag{6}$$

A *highest-weight vector* or *primary field* is an element v such that $L(n)v = 0$ for all $n > 0$; that is, v generates a highest-weight module for the Virasoro algebra. Clearly, for a highest-weight vector v, $L(1)v = L(2)v = 0$. Conversely, if $L(1)v = 0$ and $L(2)v = 0$ then using the Virasoro commutation relations, equation (2), one can show that v is a highest-weight vector.

From the creation axiom, $L(n)\mathbf{1} = \omega(n+1)\mathbf{1} = 0$, for $n \geq 0$. Thus $\mathbf{1}$ is a highest-weight vector of weight zero. From the Virasoro commutation relation and the creation axiom, $[L(0), L(-2)]\mathbf{1} = 2L(-2)\mathbf{1} = 2\omega(-1)\mathbf{1} = 2\omega$. Alternately, $[L(0), L(-2)]\mathbf{1} = L(0)L(-2)\mathbf{1} - L(-2)L(0)\mathbf{1} = L(0)\omega$. Thus $L(0)\omega = 2\omega$ and $\operatorname{wt}\omega = 2$.

We say a VOA is of *CFT-type* if the grading is truncated below by zero and the weight-zero space is spanned by $\mathbf{1}$, i.e.,

$$V = \mathbb{C}\mathbf{1} + V_1 + V_2 + \cdots .$$

We say a VOA is of *strong CFT-type* if, additionally, $L(1)V_1 = 0$. Note that in a VOA of strong CFT-type all the elements of V_1 are highest-weight vectors.

We can compute the commutator of $L(m)$ and $u(n)$ for u a weight-one element of a VOA, V, of strong CFT-type. From the commutation relation, equation (4),

$$[L(m), v(n)] = [\omega(m+1), v(n)] = \sum_{i \geq 0} \binom{m+1}{i} \big(\omega(i)v\big)(m+n+1-i).$$

Because V is of strong CFT-type, from equation (6), $\omega(i)v = 0$, for $i \geq 2$. So

$$[L(m), v(n)] = \big(L(-1)v\big)(m+n+1) + (m+1)\big(L(0)v\big)(m+n).$$

Thus, using equation (5) and part 7 of the definition of VOA,

$$\begin{aligned} [L(m), v(n)] &= (-m-n-1)v(m+n) + (m+1)v(m+n), \\ &= -nv(m+n). \end{aligned} \tag{7}$$

An *admissible V-module*, (M, Y_M) is a integrally graded vector space, $M = \bigoplus_{n=0}^{\infty} M_n$, and a linear map, $Y_M : V \to \operatorname{End} M[\![z]\!]$, $Y_M(v, z) = \sum_{n\in\mathbb{Z}} v(n)z^{-n-1}$, satisfying

1. For all u in V and v in M, there exists an integer N such that $u(n)v = 0$ for all $n > N$.
2. $Y_M(\mathbf{1}, z) = \mathrm{id}_M$.
3. $$z_0^{-1}\delta\Big(\frac{z_1 - z_2}{z_0}\Big)Y_M(u, z_1)Y_M(v, z_2) - z_0^{-1}\delta\Big(\frac{z_2 - z_1}{-z_0}\Big)Y_M(v, z_2)Y_M(u, z_1) \\ = z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big)Y_M(Y(u, z_0)v, z_2). \tag{8}$$
4. For homogeneous u, $u(m)M_n \subset M_{n+\mathrm{wt}\,u-m-1}$.

A VOA is *rational* if all of its admissible modules are completely reducible. It is *holomorphic* if it is rational and its only simple module is itself. A VOA V is C_2-*cofinite*, if the subspace generated by $\{u(-2)v | u, v \in V\}$ has finite codimension in V. We call a VOA *strongly holomorphic* if it is

1. holomorphic,
2. C_2-cofinite,
3. strong CFT-type.

We actually need only assume CFT-type as Dong and Mason have proven that if a holomorphic VOA is of CFT-type, then it is of strong CFT-type [DMa]. A *strongly rational* VOA is rational, C_2-cofinite and strong CFT-type. Thus a holomorphic, strongly rational VOA is strongly holomorphic.

It is easy to show that if a VOA, V, is of CFT-type, then V_1 is a Lie algebra, see Lemma 2.1. Dong and Mason have proven that if V is strongly rational then V_1 is *reductive* [DMb]; that is, V_1 is a Lie-algebra direct sum of abelian and semi-simple Lie algebras. Furthermore, they have shown that for $c = 24$, strongly holomorphic VOAs, V_1 is either zero, abelian or semi-simple [DMa].

A *homomorphism* of vertex-operator algebras $(V_1, Y_1, \mathbf{1}_1, \omega_1)$ and $(V_2, Y_2, \mathbf{1}_2, \omega_2)$ is a linear transformation $T : V_1 \to V_2$, such that $T\big(Y_1(u,z)v\big) = Y_2(T(u),z)T(v)$, $T(\mathbf{1}_1) = \mathbf{1}_2$ and $T(\omega_1) = \omega_2$.

2.2 The square-bracket VOA

For any VOA $(V, Y(\ ,\ ), \mathbf{1}, \omega)$, we define another VOA $(V, Y[\ ,\ ], \mathbf{1}, \tilde{\omega})$, called a *square-bracket* VOA. The underlying vector space V and the vacuum vector $\mathbf{1}$ are the same as the original *round-bracket* VOA, while the vertex-operator map is defined to be

$$Y[v,z] := Y(v, e^z - 1)e^{z \operatorname{wt} v},$$

and the conformal vector is $\tilde{\omega} = \omega - \frac{c}{24}\mathbf{1}$. The square-bracket VOA is isomorphic to the original VOA [Zhu96], [Hua92]. The square-bracket vertex operators are expanded using the notation

$$Y[v,z] =: \sum_{n \in \mathbb{Z}} v[n] z^{-n-1}.$$

The conformal element, $\tilde{\omega}$, gives rise to the square-bracket Virasoro operators, $L[n]$, via

$$Y[\tilde{\omega}, z] =: \sum_{n\in\mathbb{Z}} L[n] z^{-n-2},$$

and $L[0]$ induces the square-bracket grading on V, $V = \sum_{n\in\mathbb{Z}} V_{[n]}$, in which

$$V_{[n]} := \{v \in V | L[0]v = nv\}.$$

We say $v \in V_{[n]}$ has square-bracket weight n and write $[\mathrm{wt}]v = n$. Because they are isomorphic, if a VOA is of CFT-type, then its square-bracket VOA is of CFT-type; however, in general $V_{[n]} \neq V_n$. But, because the vacuum vector is the same for both VOAs, we do have $V_{[0]} = \mathbb{C}\mathbf{1} = V_0$.

We will frequently need to convert from square-bracket operators to round bracket operators. We use the following equation for round-bracket homogeneous v [Zhu96], [DLM00, eq. 2.13],

$$v[n] = \mathrm{Res}_z\, Y(v, z)\big(\log(1 + z)\big)^n (1 + z)^{\mathrm{wt}\, v-1}. \tag{9}$$

From the definitions,

$$L[n] = \tilde{\omega}[n + 1] = \omega[n + 1] - \frac{c}{24}\mathbf{1}[n + 1] = \begin{cases} \omega[n + 1] - \frac{c}{24}\,\mathrm{id} & \text{if } n = -2, \\ \omega[n + 1] & \text{otherwise.} \end{cases} \tag{10}$$

In particular, $L[0] = \omega[1]$, and applying equation (9) with $\mathrm{wt}\,\omega = 2$ gives

$$L[0] = L(0) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n + 1)} L(n). \tag{11}$$

We can also convert from round brackets to square brackets. Since

$$\frac{d}{dz}(e^z - 1)^{m-n} = (m - n)(e^z - 1)^{m-n-1}e^z,$$

we have

$$\operatorname{Res}_z (e^z - 1)^{m-n-1} e^z = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Thus $v(m) = \operatorname{Res}_z \big(Y(v, e^z - 1)(e^z - 1)^m e^z \big)$; and if v is round-bracket homogeneous, from the definition of $Y[v, z]$ we see that

$$v(m) = \operatorname{Res}_z Y[v, z](e^z - 1)^m e^{z(1 - \operatorname{wt} v)}. \tag{12}$$

In particular,

$$L(0) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(i+1)!} L[i]. \tag{13}$$

Let $v \in V_{[N]}$ and write $v = v_m + v_{m-1} + \cdots + v_{n_0}$, with $v_i \in V_i$ and $v_m \neq 0$. We have

$$L[0]v = Nv = Nv_m + Nv_{m-1} + \cdots + Nv_{n_0}.$$

Equation (6) implies that $L(i) : V_j \to V_{j-i}$. So equation (11) implies that

$$\begin{aligned} L[0]v &= L(0)v_m + \text{terms in } V_i,\ i < m, \\ &= mv_m + \text{terms in } V_i,\ i < m. \end{aligned}$$

Projecting both of the above equations into V_m yields $Nv_m = mv_m$. Thus $m = N$ and $V_{[N]} \subseteq \bigoplus_{n \le N} V_n$. Using equation (13) in the same way, we see that $V_N \subseteq \bigoplus_{n \le N} V_{[n]}$. So

$$\bigoplus_{n \le N} V_{[n]} = \bigoplus_{n \le N} V_n.$$

2.3 The graded trace and modular forms

For homogeneous v, define the *zero mode* of v to be $o(v) := v(\operatorname{wt} v - 1)$. Extend o linearly to all of V. Equation (6) implies that $o(v)$ preserves the grading of V, i.e., $o(v) : V_n \to V_n$. Define the *graded trace* of v or *one-point correlation function* of v to be

$$Z(v,q) := q^{-c/24} \sum_{n\in\mathbb{Z}} \operatorname{tr}\big|_{V_n} o(v) q^n.$$

We can view q as a complex variable of modulus less than one. Zhu proved that for C_2-cofinite VOAs, $Z(v,q)$ converges to a meromorphic function, which is holomorphic for $q \neq 0$ [Zhu96]. Define τ in the complex upper half plane by $q = e^{2\pi i\tau}$, note that $q = 0$ corresponds to $\tau = \infty$ and let $Z(v,\tau) = Z(v,q)$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of the modular group, $PSL_2(\mathbb{Z})$, and define an action of g on $Z(v,\tau)$ via

$$g \circ Z(v,\tau) := Z\Big(v, \frac{a\tau + b}{c\tau + d}\Big).$$

Zhu also showed that for a holomorphic C_2-cofinite VOA and a square-bracket homogeneous vector v, $Z(v,q)$ is a meromorphic modular form of weight $k = [\mathrm{wt}]v$ [Zhu96]; that is,

$$g \circ Z(v,\tau) = \chi(g)(c\tau + d)^{[\mathrm{wt}]v} Z(v,\tau).$$

Here χ is a group character of the modular group, i.e., χ is a group homomorphism from $PSL_2(\mathbb{Z})$ to $\mathbb{C}^*$. It depends on the VOA but not the element v. Since $\mathbb{C}^*$ is abelian, and the abelianization of $PSL_2(\mathbb{Z})$ is a cyclic group generated by the coset of the commutator subgroup containing $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; to determine χ it suffices to check $T \circ Z(v,\tau)$. Because $T : \tau \mapsto \tau + 1$, T fixes $q = e^{2\pi i\tau}$. If $24|c$, then the q-expansion of $Z(v,\tau)$ contains only integral powers of q and therefore is fixed by T. Thus if $24|c$, then the character, $\chi(g)$, is trivial, and we say that there is no character.

For VOAs of CFT-type with central charge 24,

$$Z(v,\tau) = \sum_{n=0}^{\infty} \operatorname{tr}\big|_{V_n} o(v) q^{n-1}.$$

So $Z(v,\tau)$ is holomorphic at $\tau = \infty$, $q = 0$, if and only if $\operatorname{tr}\big|_{V_0} o(v) = 0$. Assume that V is a strongly holomorphic $c = 24$ VOA and that v is a square-bracket homogeneous vector with $[\mathrm{wt}]v = k$, such that $\mathrm{tr}\big|_{V_0} o(v) = 0$, then $Z(v, \tau)$ is a holomorphic modular form of weight k with no character.

The space of holomorphic modular forms without characters is well known; see [Ser73]. It is a commutative algebra, in that the sum of two forms of weight k is a form of weight k and the product of a weight-k form and a weight-ℓ form is a form of weight $k + \ell$.

Denote the space of holomorphic forms of weight k by M_k. For each even integer $k \geq 4$, M_k has an element $E_k(\tau)$, called the Eisenstein series of weight k and defined

$$E_k(\tau) := -\frac{B_k}{k!} + \frac{2}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \tag{14}$$

Here B_k is the kth Bernoulli number defined by $\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$, and $\sigma_\ell(n) := \sum_{d|n} d^\ell$. For $k = 4$, 6, 8 or 10, M_k is one-dimensional and generated by $E_k(\tau)$. Thus, for those k, if we can find a square-bracket-weight-k element, v, with $\mathrm{tr}\big|_{V_0} o(v) = 0$ and $\mathrm{tr}\big|_{V_1} o(v) \neq 0$, then some multiple of v has graded trace equal to $E_k(\tau)$.

There is a differential operator $\delta_k : M_k \to M_{k+2}$ defined $\delta_k = \frac{d}{dq} + kE_2(\tau)$, where $E_2(\tau)$ is defined by equation (14), but is *not* a modular form, see [Lan76, ch.X §5]. Define a δ-ideal to be an ideal in the algebra of holomorphic modular forms that is closed under the δ_k's. Because $\delta_4 E_4 = 14E_6$ and the algebra of holomorphic modular forms is generated by E_4 and E_6, the δ-ideal generated by E_4 is the space of all holomorphic modular forms.

2.4 Affine Lie algebras

First we prove that for vertex operator algebras of CFT-type, V_1 is a Lie algebra, as we claimed earlier.

Lemma 2.1. *Assume that V is a VOA of CFT-type. Then V_1 is an Lie algebra with bracket $[u, v] = u(0)v$ and invariant symmetric bilinear form $\langle\ |\ \rangle$ defined by $u(1)v = \langle u|v\rangle \mathbf{1}$.*

Proof. The linearity of [,] is clear. To show that it is alternating we use skew-symmetry [FHL93, eq. 2.3.19],

$$\begin{aligned}\mathrm{Res}_z\, Y(u,z)v &= \mathrm{Res}_z\, e^{zL(-1)}Y(v,-z)u\\ u(0)v &= -v(0)u + L(-1)v(1)u + \cdots + (-1)^{i+1}\frac{1}{i!}L(-1)^i v(i)u + \cdots .\end{aligned}$$

Since $\mathrm{wt}\, v(i)u = 1-i$ and V is of CFT type, $v(i)u = 0$ for $i \geq 2$. Moreover, $v(1)u \in V_0 = \mathbb{C}\mathbf{1}$ and $L(-1)\mathbf{1} = \omega(0)\mathbf{1} = 0$, so $L(-1)v(1)u = 0$ and the above equation becomes $u(0)v = -v(0)u$, that is $[u,v] = -[v,u]$.

By taking $\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}$ of both sides of the VOA Jacobi identity, equation (1), we reduce it to $u(0)v(0) - v(0)u(0) = (u(0)v)(0)$. Thus $[u,[v,w]] - [v,[u,w]] - [[u,v],w] = 0$ and [,] satisfies the Lie-algebra Jacobi identity.

The bilinearity of $\langle\ |\ \rangle$ is clear. The proof that it is symmetric is very similar to the proof that the Lie bracket is alternating.

$$\begin{aligned}\mathrm{Res}_z\, zY(u,z)v &= \mathrm{Res}_z\, ze^{zL(-1)}Y(v,-z)u\\ u(1)v &= v(1)u - L(-1)v(2)u + \cdots + (-1)^i\frac{1}{i!}L(-1)^i v(i+1)u + \cdots\\ u(1)v &= v(1)u\end{aligned}$$

To show invariance we use associativity, equation (3).

$$\begin{aligned}\langle u|[v,w]\rangle\mathbf{1} &= \langle [v,w]|u\rangle\mathbf{1}\\ &= \big(v(0)w\big)(1)u\\ &= v(0)w(1)u - w(1)v(0)u\\ &= 0 - \langle w|[v,u]\rangle\mathbf{1}\\ &= \langle [u,v]|w\rangle\mathbf{1}\end{aligned}$$

□

Let $\mathfrak{g}$ be a Lie algebra. The (untwisted) affine Lie algebra $A(\mathfrak{g})$ is defined to be

$$A(\mathfrak{g}) := \mathfrak{g} \otimes \mathbb{C}[t^{-1}, t] \oplus \mathbb{C}c.$$

Denoting $a \otimes t^n$ by $a(n)$, the Lie brackets in $A(\mathfrak{g})$ are

$$\begin{aligned} [a(m), b(n)] &:= [a, b](m+n) + m\langle a|b\rangle c\delta_{m+n,0}, \\ [c, A(\mathfrak{g})] &:= 0, \end{aligned} \tag{15}$$

for all $m, n \in \mathbb{Z}$. Define

$$a(z) := \sum_{n\in\mathbb{Z}} a(n)z^{-n-1}.$$

We rewrite the above commutation relations, equation (15), in a single equation as

$$[a(z_1), b(z_2)] = [a, b](z_2)z_2^{-1}\delta\Big(\frac{z_1}{z_2}\Big) - \langle a|b\rangle cz_2^{-1}\frac{\partial}{\partial z_1}\delta\Big(\frac{z_1}{z_2}\Big).$$

Proposition 2.2. *Let V be a VOA of CFT type. The operators $u(n) \in \mathrm{End}(V)$, $u \in V_1$ are isomorphic to the affine Lie algebra $A(V_1)$.*

Proof. From the previous lemma, V_1 is a finite-dimensional Lie algebra with an invariant bilinear form. Consider the map

$$\begin{aligned} F : A(V_1) &\to \mathrm{End}(V), \\ u(z) &\mapsto Y(u, z) \\ c &\mapsto \mathrm{id}\,. \end{aligned}$$

Take Res_{z_0} of both sides of the VOA Jacobi identity.

$$\begin{aligned} [Y(u, z_1), Y(v, z_2)] &= \mathrm{Res}_{z_0}\, z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big)Y(Y(u, z_0)v, z_2) \\ &= \mathrm{Res}_{z_0} \sum_{\substack{i,n,m\in\mathbb{Z} \\ j\in\mathbb{N}}} z_2^{-1}(-1)^j\binom{i}{j} z_0^j z_1^{i-j} z_2^{-i}(u(n)v)(m)z_0^{-n-1}z_2^{-m-1} \end{aligned}$$

$$[Y(u,z_1),Y(v,z_2)] = \sum_{\substack{i,m\in\mathbb{Z}\\ j\in\mathbb{N}}} (-1)^j \binom{i}{j} (u(j)v)(m) z_1^{i-j} z_2^{-i-m-2}$$

Since u and v are weight one, $u(j)v = 0$ for $j \geq 2$. Thus we have only the $j = 0$ and $j = 1$ terms. The $j = 1$ term, $(u(1)v)(m) = \langle u|v\rangle \mathbf{1}(m)$, equals $\langle u|v\rangle \,\mathrm{id}$ for $m = -1$ and zero for all other m. Thus

$$\begin{aligned}[Y(u,z_1),Y(v,z_2)] &= \sum_{i,m\in\mathbb{Z}} [u,v](m) z_2^{-m-i-2} z_1^{i} + \sum_{i\in\mathbb{Z}} -i\langle u|v\rangle \,\mathrm{id}\, z_1^{i-1} z_2^{-i-1}, \\ &= z_2^{-1} Y([u,v],z_2)\delta\Big(\frac{z_1}{z_2}\Big) - \langle u|v\rangle \,\mathrm{id}\, z_2^{-1} \frac{\partial}{\partial z_1}\delta\Big(\frac{z_1}{z_2}\Big).\end{aligned}$$

Additionally, id commutes with all elements of $\mathrm{End}(V)$. Therefore F is a Lie algebra homomorphism. From the creation axiom, if $Y(u,z) = Y(v,z)$, then $u = v$, so F is injective. $\square$

3 Main Theorem

Theorem 3.1. *Let V be a strongly holomorphic $c = 24$ VOA with V_1 non-abelian and non-zero. For each modular form f there exists a state v in V such that the graded trace of v is f.*

Note that by modular form, we mean holomorphic modular form with no character, and that, using Dong and Mason's result, V_1 non-abelian and non-zero is equivalent to V_1 semisimple [DMa].

A key ingredient in the proof of this theorem is the following proposition of Dong and Mason [DM00, Prop. 2].

Proposition 3.2. *Suppose that V is a strongly holomorphic $c = 24$ VOA and w is a highest-weight vector of positive weight with graded trace $Z(w,\tau) = f$. The space of $Z(v,\tau)$ for v in the Virasoro module generated by w equals the δ-ideal generated by f.*

Dong and Mason only state their proposition for $V^\natural$; however, the proof holds for the larger class of VOAs. The δ-ideal generated by E_4 is the space of all modular forms. Thus, to prove Theorem 3.1, we need to find a highest-weight vector with graded trace E_4.

We will actually show that there exists an element of $V_{[4]}$ with graded trace E_4. By the following proposition, this is enough.

Proposition 3.3. *Let V be a strongly holomorphic $c = 24$ VOA. Let $v \in V_{[4]}$. If $Z(v, \tau)$ is holomorphic, then there exists a highest-weight vector w such that $Z(w, \tau) = Z(v, \tau)$.*

4 Proof of Proposition 3.3

Propositions 3.2 and 3.3 invoke highest-weight vectors. A priori, this means round-bracket highest-weight vectors; however, by the following lemma, these are the same as square-bracket highest-weight vectors.

Lemma 4.1. *The set of square-bracket highest-weight vectors v, with $[\mathrm{wt}]v = n$ equals the set of round-bracket highest-weight vectors u, with $\mathrm{wt}\, u = n$.*

Proof. Let u be a round-bracket highest weight vector of weight n, i.e., $L(n)u = 0$ for $n > 0$ and $L(0)u = nu$. Equation (11) implies that $L[0]u = nu$, that is $u \in V_{[n]}$. From equation (9), we have

$$\begin{aligned} L[1] &= L(1) - \frac{1}{12}L(3) + \frac{1}{12}L(4) + \cdots , \\ L[2] &= L(2) - \frac{1}{2}L(3) + \frac{1}{4}L(4) + \cdots . \end{aligned}$$

So $L[1]u = L[2]u = 0$ and u is a square-bracket highest-weight vector. The converse is proved the same way, using equation (13) and the square-bracket expansions:

$$\begin{aligned} L(1) &= L[1] + \frac{1}{12}L[3] + \frac{1}{360}L[5] + \ldots , \\ L(2) &= L[2] + \frac{1}{2}L[3] + \frac{1}{4}L[4] + \frac{1}{12}L[5] \ldots , \end{aligned}$$

which follow from equation (12). $\square$

Thus to prove Proposition 3.3 we will show that if $Z(v,\tau)$ with $v \in V_{[4]}$ is holomorphic, then there exists a square-bracket highest-weight vector, w with $Z(w,\tau) = Z(v,\tau)$. We use the following characterization of those v with holomorphic graded traces.

Lemma 4.2. *Let $v \in V$ have decomposition $v = v_0 + v_1 + \cdots + v_n$, with $v_i \in V_i$. Then $Z(v,\tau)$ is holomorphic if and only if $v_0 = 0$.*

Proof. The graded trace is holomorphic, except for a possible pole at $q = 0$. For $c = 24$ VOAs, the expansion of $Z(v,\tau)$ about $q = 0$ is $q^{-1}\sum_{n\geq 0} \operatorname{tr}\big|_{V_n} o(v)q^n$. So $Z(v,\tau)$ is holomorphic if and only if $\operatorname{tr}\big|_{V_0} o(v) = 0$. We have $o(v) = v_0(-1) + v_1(0) + \cdots + v_n(n-1)$. Because V is of CFT-type, $V_0 = \mathbb{C}\mathbf{1}$, and we can compute the trace of $o(v)$ on V_0 by computing its action on the vacuum. From the creation axiom, $v_0(-1)\mathbf{1} = v_0$ and $v_i(i-1)\mathbf{1} = 0$ for $i > 0$. Thus $o(v)\mathbf{1} = v_0$. Let $k \in \mathbb{C}$ be defined so that $v_0 = k\mathbf{1}$. We have $\operatorname{tr}\big|_{V_0} o(v) = k$, and see that $\operatorname{tr}\big|_{V_0} o(v) = 0$ if and only if $v_0 = 0$. $\square$

We also make use of the symmetric invariant bilinear form on on V. A bilinear form $(\, , \,)$ on V is said to be *invariant* if

$$\big(Y(u,z)v, w\big) = \big(v, Y(e^{zL(1)}(-z^{-2})^{L(0)}u, z^{-1})w\big), \tag{16}$$

for all $u, v, w \in V$ [FHL93]. Li proved that the space of invariant forms on V is isomorphic to $\big(\frac{V_0}{L(1)V_1}\big)^*$ [Li94]. Since V is strong CFT-type, $\big(\frac{V_0}{L(1)V_1}\big)^*$ is one-dimensional, and the invariant bilinear form on V is unique up to a scalar. Henceforth, we fix $(\, , \,)$ to be the unique invariant form on V so that $(\mathbf{1},\mathbf{1}) = 1$. Because V is simple, $(\, , \,)$ is non-degenerate.

Let V be of strong CFT-type, and let u and v be in V_1. Then using the invariance of the VOA form $(\, , \,)$, equation (16),

$$(u,v) = \operatorname{Res}_z z^{-1}(Y(u,z)\mathbf{1}, v),$$

$$
\begin{aligned}
(u,v) &= -\operatorname{Res}_z z^{-3}(\mathbf{1}, Y(u,z^{-1})v),\\
&= -(\mathbf{1}, u(1)v).
\end{aligned}
$$

Recall that we defined the bilinear form $\langle\ |\ \rangle$ on V_1 by $u(1)v = \langle u|v\rangle\mathbf{1}$. So the restriction of $(\ ,\ )$ to V_1 equals $-\langle\ |\ \rangle$.

Harada and Lam showed that $V_{[4]} = \operatorname{Ker} L[1]\big|_{V_{[4]}} \oplus L[-1]V_{[3]}$ and $V_{[4]} = \operatorname{Ker} L[2]\big|_{V_{[4]}} \oplus L[-2]V_{[2]}$, both sums orthogonal with respect to a *square-bracket* invariant bilinear form on V [HL95]. Because the for is non-degenerate, it follows that

$$
V_{[4]} = (\operatorname{Ker} L[1] \cap \operatorname{Ker} L[2]) \oplus (L[-1]V_{[3]} + L[-2]V_{[2]}).
$$

We will show that if there exists an element $v \in V_{[4]}$ such that $Z(v,\tau)$ is holomorphic, then there must exist an element of $\operatorname{Ker} L[1] \cap \operatorname{Ker} L[2]$ with the same graded trace. Because the space of highest-weight vectors in $V_{[4]}$ is $\operatorname{Ker} L[1] \cap \operatorname{Ker} L[2]$, this will prove Proposition 3.3.

Suppose that $a = b+c$, with $a \in V_{[4]}$, $b \in \operatorname{Ker} L[1] \cap \operatorname{Ker} L[2]$ and $c \in L[-1]V_{[3]} + L[-2]V_{[2]}$, and that $Z(a,z)$ is holomorphic. By Lemma 4.1, a highest-weight vector in $V_{[4]}$ is in V_4. So from the previous lemma, $Z(b,z)$ is holomorphic. Therefore $Z(c,z)$ is also holomorphic. Let $c = L[-1]u + L[-2]v$ with $u \in V_{[3]}$ and $v \in V_{[2]}$. Because $Z(L[-1]u,\tau) = 0$ [Zhu96] and [DLM00], $Z(c,\tau) = Z(L[-2]v,\tau)$. If we prove that $Z(L[-2]v,\tau) = 0$, then $Z(a,\tau) = Z(b,\tau)$ and Proposition 3.3 follows. Thus we are reduced to the following claim.

Claim 4.3. *Let $v \in V_{[2]}$. If $Z(L[-2]v,\tau)$ is holomorphic, then it is identically zero.*

Proof. Write v in terms of round brackets, $v = v_2 + v_1 + v_0$, $v_i \in V_i$. As $c = 24$, in the square bracket VOA, the conformal element is $\tilde{\omega} = \omega - \mathbf{1}$. Using equations (10) and (9), we compute

$$
L[-2] = -\operatorname{id} + L(-2) + \frac{3}{2}L(-1) + \frac{5}{12}L(0) - \frac{1}{24}L(1) + \frac{11}{720}L(2) + \cdots .
$$

So the round-bracket-weight-zero part of $L[-2]v$ is $-v_0 + \frac{5}{12}L(0)v_0 - \frac{1}{24}L(1)v_1 + \frac{11}{720}L(2)v_2$.

Furthermore, $L(0)v_0 = 0$ and as V is strong CFT-type $L(1)v_1 = 0$. So the round-bracket-weight-zero part of $L[-2]v$ is $-v_0 + \frac{11}{720}L(2)v_2$. From Lemma 4.2, as $Z(L[-2]v, \tau)$ is holomorphic, $-v_0 + \frac{11}{720}L(2)v_2 = 0$. Because $L[2] = L(2) - (1/2)L(3) + \cdots$, we have $L[2]v = L(2)v_2$. Thus

$$L[2]v = \frac{720}{11}v_0. \tag{17}$$

We now turn to the following equation from Zhu [Zhu96], which holds for C_2-cofinite VOAs – see also [DLM00, eq. 5.8]. For $v \in V_{[2]}$,

$$Z(L[-2]v, \tau) = \partial Z(v, \tau) + E_4(\tau)Z(L[2]v, \tau). \tag{18}$$

Here $\partial = \delta_k = q\frac{d}{dq} + kE_2(\tau)$, where k is the weight of the modular form to which ∂ is applied. Note that if f is holomorphic then ∂f is also holomorphic. Let ℓ be defined so that $v_0 = \ell\mathbf{1}$, then, from equations (17) and (18),

$$\begin{aligned} Z(L[-2]v, \tau) &= \partial Z(v_2 + v_1, \tau) + \partial Z(v_0, \tau) + \frac{720}{11}E_4(\tau)Z(v_0, \tau), \\ &= 0q^{-1} + \cdots + -\ell q^{-1} + \cdots + \frac{720}{11}\Big(\frac{1}{720} + \frac{1}{3}\sum_{n>0}\sigma_3(n)q^n\Big)(\ell q^{-1} + \cdots), \\ &= -\frac{10\ell}{11}q^{-1} + \cdots . \end{aligned}$$

Because $Z(L[-2]v, \tau)$ is holomorphic, $\ell = 0$, that is $v_0 = 0$, and from equation (17), $L[2]v = 0$. Additionally, from Lemma 4.2, $v_0 = 0$ implies that $Z(v, \tau)$ is holomorphic. Because $v \in V_{[2]}$, $Z(v, \tau)$ has weight two. There are no non-zero weight-two holomorphic modular forms, so $Z(v, \tau) = 0$. Referring back to equation (18), $Z(L[-2]v, \tau) = 0$. □

5 General Graded Traces

Recall that any VOA is isomorphic to its square-bracket VOA, hence $\dim V_{[n]} = \dim V_n$; however, in general, $V_{[n]} \neq V_n$. Let V be a VOA of strong CFT-type, and let $v \in V_1$. From equation (11),

$$L[0]v = L(0)v + \frac{1}{2}L(1)v = L(0)v = v.$$

Thus $v \in V_{[1]}$, so $V_1 \subseteq V_{[1]}$, and because $\dim V_{[1]} = \dim V_1$, $V_{[1]} = V_1$. Thus the affine Lie algebra derived from the square-bracket VOA, $A(V_{[1]}) := \{v[n] | v \in V_{[1]}\}$, equals $\{v[n] | v \in V_1\}$. We can apply the endomorphisms of $A(V_{[1]})$ to V_1 repeatedly to generate elements of V. Define $A(V_{[1]})V_1 := \mathrm{Span}\{u_k[n_k]u_{k-1}[n_{k-1}] \cdots u_1[n_1]v | u_i, v \in V_1, n_i \in \mathbb{Z}, k \geq 0\}$. For V_1 semisimple, we have the following characterization of $A(V_{[1]})V_1$.

Proposition 5.1. *Let V be a VOA of strong CFT-type such that V_1 is a semisimple Lie algebra, then* $A(V_{[1]})V_1 = \mathrm{Span}\left(\mathbf{1} \cup \{u_k[-1]u_{k-1}[-1] \cdots u_1[-1]v | u_i, v \in V_1, k \geq 0\}\right)$.

Proof. Let $u_k[n_k]u_{k-1}[n_{k-1}] \cdots u_1[n_1]v$, be a generic element of $A(V_{[1]})V_1$. Use the affine Lie algebra commutation relations to write w as a linear combination of $u_k[n_k]u_{k-1}[n_{k-1}] \cdots u_1[n_1]v$ such that $n_k \leq n_{k-1} \leq \cdots \leq n_1$.

For $n_1 \geq 2$, $u_1[n_1]v = 0$, because $V_{[1-n]} = 0$. From equation (9), for $v \in V_1$,

$$\begin{aligned} &v[0] = v(0), \\ &v[1] = v(1) - \frac{1}{2}v(2) + \frac{1}{3}v(3) + \cdots . \end{aligned}$$

As an operator on $V_{[1]} = V_1$, $v(n) = 0$ for $n \geq 2$; so $v[1] = v(1)$. Suppose that $n_1 = 1$. Then $u_1[1]v = u_1(1)v = \langle u_1 | v \rangle \mathbf{1}$. Because if $n_2 = 0$ or 1 $u_2[n_2]\mathbf{1} = 0$, only the terms with all the other $n_i \leq -1$ survive. Now, suppose that $n_1 = 0$. Then $u_1[0]v = u_1(0)v = [u_1, v] \in V_1$. By repeating this for all the i such that $n_i = 0$, we can write our term as $u_k[n_k]u_{k-1}[n_{k-1}] \cdots u_j[n_j]w$, with $n_j \leq -1$ and $w \in V_1$. Thus our original element is written as a linear combination of $u_k[n_k]u_{k-1}[n_{k-1}] \cdots u_1[n_1]v$ with $n_i \leq -1$, $u_i \in V_1$ and $v \in V_1$ or $v = \mathbf{1}$.

Suppose some $n_i \leq -2$. Because V_1 is semisimple there exist u_{i_1} and u_{i_2} in V_1 such that $u_i = [u_{i_1}, u_{i_2}]$. We can now use the affine Lie algebra commutation relations to show that

$$u_i[n_i] = [u_{i_1}, u_{i_2}][n_i] = [u_{i_1}[n_i + 1], u_{i_2}[-1]] = u_{i_1}[n_i + 1]u_{i_2}[-1] - u_{i_2}[-1]u_{i_1}[n_i + 1].$$

If $n_i + 1 \le -2$ we repeat this process with $u_{i_1}[n_i + 1]$. In this manner we can rewrite w again, this time as a linear combination of $u_k[-1]u_{k-1}[-1] \cdots u_1[-1]v$ with $u_i \in V_1$ and $v \in V_1$ or $v = \mathbf{1}$, $k \ge 0$. Because $u_1[-1]\mathbf{1} = u_1$, the result follows. □

Therefore for V_1 semisimple, the intersection of $A(V_{[1]})V_1$ and $V_{[4]}$ is spanned by elements $a[-1]b[-1]c[-1]d$ with a, b, c and d in V_1 We want to find the traces of $o(a[-1]b[-1]c[-1]d)$ on V_0 and V_1.

The following identites are used repeatedly in the computations in this section. The affine Lie algebra commutation relations, equation (15): For all u and v in V_1,

$$[u(m), v(n)] = [u, v](m + n) + \langle u|v\rangle m\delta_{m+n,0}\,\mathrm{id};$$

and associativity, equation (3): For all u and v in V,

$$\big(u(r)v\big)(q) = \sum_{i\ge 0}(-1)^i\binom{r}{i}\big(u(r-i)v(q+i) - (-1)^r v(r+q-i)u(i)\big).$$

A priori, associativity yields an infinite sum; however, when applied to a specific homogeneous space, all but finitely many terms are zero. Additionally, for any v, w in V_1, $u(n)v(1)w = u(n)\langle v|w\rangle\mathbf{1}$. By the creation axiom, $u(n)\mathbf{1} = 0$ for $n \ge 0$. Hence, as an operator on V_1,

$$u(n)v(1) = 0 \tag{19}$$

for all u in V, v in V_1 and $n \ge 0$. Also useful is the Lie algebra identity

$$([a, c] : [b, d]) = ([a, b] : [c, d]) + ([a, d] : [b, c]), \tag{20}$$

for any a, b, c and d elements of a Lie algebra with a symmetric invariant bilinear form $(\ :\ )$. This identity follows easily from the Lie-algebra Jacobi identity.

Because $o(v) = v(\mathrm{wt}\, v - 1)$, in order to find the zero mode of $a[-1]b[-1]c[-1]d$, we expand it into round-bracket homogeneous terms and take the zero mode of each term. Thus, in order to compute the trace of $o(a[-1]b[-1]c[-1]d)$ on V_1, we need the traces on V_1 of the zero-modes of assorted round-bracket homogeneous vectors. The traces that we use are given in the following lemma.

Lemma 5.2. *Let V be a strongly rational VOA, and let u, v, w and x be elements of V_1.*

$$\operatorname{tr}\Big|_{V_1} u(0) = 0, \tag{21}$$

$$\operatorname{tr}\Big|_{V_1} (u(-1)v)(1) = \kappa(u,v) + 2\langle u|v\rangle, \tag{22}$$

$$\operatorname{tr}\Big|_{V_1} (u(0)v(-1)w)(1) = 0, \tag{23}$$

$$\operatorname{tr}\Big|_{V_1} (u(-1)v(-1)w)(2) = -\langle [u,v]|w\rangle + \operatorname{tr}\Big|_{V_1} w(0)v(0)u(0), \tag{24}$$

$$\operatorname{tr}\Big|_{V_1} (u(-1)v(-1)w(-1)x)(3) = -\langle [u,v]|[w,x]\rangle - 2\langle [u,x]|[v,w]\rangle + \operatorname{tr}\Big|_{V_1} x(0)w(0)v(0)u(0). \tag{25}$$

Proof. Equation (21) says that the adjoint action of any element of V_1 has zero trace. Dong and Mason showed that for strongly rational VOAs V_1 is reductive, i.e., there exist V_s, semi-simple, and V_a, abelian, such that $V_1 = V_s \oplus V_a$ is a Lie-algebra decomposition [DMb]. If v is in V_a, then $v(0)w = [v,w] = 0$ for all $w \in V_1$. So $\operatorname{tr}|_{V_1} v(0) = 0$. The commutator of V_1 is V_s. So, if v is in V_s, there exist $x, y \in V_1$ such that $v = [x,y]$. Thus $v(0) = [x,y](0) = x(0)y(0) - y(0)x(0)$, and because $\operatorname{tr}|_{V_1} x(0)y(0) = \operatorname{tr}|_{V_1} y(0)x(0)$, we have that $\operatorname{tr}|_{V_1} v(0) = 0$. The trace is linear so these two cases suffice to prove equation (21).

Let w be in V_1; $u(-1)v(1)w = u(-1)\langle v|w\rangle \mathbf{1} = \langle v|w\rangle u$. Hence

$$\operatorname{tr}\Big|_{V_1} u(-1)v(1) = \langle v|u\rangle = \langle u|v\rangle. \tag{26}$$

Furthermore, from the definition of the Killing form κ,

$$\operatorname{tr}\Big|_{V_1} u(0)v(0) = \kappa(u,v). \tag{27}$$

To prove equation (22), we apply associativity,

$$(u(-1)v)(1) = \sum_{i\geq 0}(-1)^i\binom{-1}{i}\big(u(-1-i)v(1+i) + v(-i)u(i)\big).$$

Equation (6) implies that $\mathrm{wt}(x(i)y) = 1-i$. Because V is of CFT-type, $x(i)y = 0$ for $i \geq 2$. So considering the action of $(u(-1)v)(1)$ on V_1, we can ignore all terms $u(-1-i)v(1+i)$, $i \geq 1$ and $v(-i)u(i)$, $i \geq 2$. Therefore, as an operator on V_1,

$$(u(-1)v)(1) = u(-1)v(1) + v(0)u(0) + v(-1)u(1),$$

and using equations (26) and (27) we get equation (22).

Using the affine-Lie-algebra commutation relations, equation (15),

$$\begin{aligned} u(0)v(-1)w &= [u(0), v(-1)]w + v(-1)u(0)w, \\ &= [u,v](-1)w + v(-1)[u,w]. \end{aligned}$$

Using equation (22) and the invariance of $\langle\ |\ \rangle$ and $\kappa(\ ,\ )$

$$\begin{aligned} \mathrm{tr}\Big|_{V_1}(u(0)v(-1)w)(1) &= \mathrm{tr}\Big|_{V_1}([u,v](-1)w)(1) + \mathrm{tr}\Big|_{V_1}(v(-1)[u,w])(1), \\ &= \kappa([u,v],w) + 2\langle [u,v]|w\rangle + \kappa(v,[u,w]) + 2\langle v|[u,w]\rangle, \\ &= \kappa([u,v],w) + 2\langle [u,v]|w\rangle + \kappa([v,u],w) + 2\langle [v,u]|w\rangle, \\ &= \kappa([u,v],w) + 2\langle [u,v]|w\rangle - \kappa([u,v],w) - 2\langle [u,v]|w\rangle, \\ &= 0, \end{aligned}$$

which is equation (23).

We expand $(u(-1)v(-1)w)(2)$ as an operator on V_1 by applying associativity twice. We truncate those terms which act as zero on V_1, either from equation (19) or because they map

V_1 to negative weight spaces, as in the proof of equation (22).

$$\begin{aligned}(u(-1)v(-1)w)(2) &= u(-1)(v(-1)w)(2) + (v(-1)w)(1)u(0),\\ &= u(-1)w(1)v(0) + v(-1)w(1)u(0) + w(0)v(0)u(0) + w(-1)v(1)u(0).\end{aligned}$$

Let a, b, c and v be in V_1. We compute $a(-1)b(1)c(0)v = a(-1)b(1)[c, v] = a(-1)\langle b|[c, v]\rangle \mathbf{1} = \langle b|[c, v]\rangle a$. Therefore

$$\operatorname{tr}\Big|_{V_1} a(-1)b(1)c(0) = \langle b|[c, a]\rangle.$$

Putting this together with the expansion of $(u(-1)v(-1)w)(2)$, we get

$$\begin{aligned}\operatorname{tr}\Big|_{V_1}(u(-1)v(-1)w)(2) &= \langle w|[v, u]\rangle + \langle w|[u, v]\rangle + \operatorname{tr}\Big|_{V_1} w(0)v(0)u(0) + \langle v|[u, w]\rangle\\ &= -\langle [u, v]|w\rangle + \operatorname{tr}\Big|_{V_1} w(0)v(0)u(0)),\end{aligned}$$

which is equation (24).

To expand $(u(-1)v(-1)w(-1)x)(3)$ as on operator on V_1, we use associativity three times and truncate as above;

$$\begin{aligned}\big(u(-1)(v(-1)w(-1)x)\big)(3) &= u(-1)\big(v(-1)(w(-1)x)\big)(3) + \big(v(-1)(w(-1)x)\big)(2)u(0),\\ &= u(-1)(w(-1)x)(2)v(0) + v(-1)(w(-1)x)(2)u(0)\\ &\quad + (w(-1)x)(1)v(0)u(0),\\ &= u(-1)x(1)w(0)v(0) + v(-1)x(1)w(0)u(0) + w(-1)x(1)v(0)u(0)\\ &\quad + x(0)w(0)v(0)u(0) + x(-1)w(1)v(0)u(0).\end{aligned}$$

Now let a, b, c, d and v be in V_1; $a(-1)b(1)c(0)d(0)v = a(-1)b(1)[c, [d, v]] = \langle b|[c, [d, v]]\rangle a$. Thus

$$\operatorname{tr}\Big|_{V_1} a(-1)b(1)c(0)d(0) = \langle [b, c]|[d, a]\rangle.$$

Using this, the expansion of $(u(-1)v(-1)w(-1)x)(3)$ and equation (20), we compute

$$\begin{aligned}\operatorname{tr}\Big|_{V_1}(u(-1)v(-1)w(-1)x)(3) &= \langle [x,w]|[v,u]\rangle + \langle [x,w]|[u,v]\rangle + \langle [x,v]|[u,w]\rangle \\ &\quad + \operatorname{tr}\Big|_{V_1} x(0)w(0)v(0)u(0) + \langle [w,v]|[u,x]\rangle, \\ &= -\langle [u,v]|[w,x]\rangle - 2\langle [u,x]|[v,w]\rangle + \operatorname{tr}\Big|_{V_1} x(0)w(0)v(0)u(0).\end{aligned}$$

This is equation (25). $\square$

Proposition 5.3. *Let V be a strongly rational VOA. Let a, b, c and d be elements of V_1 and n be the dimension of V_1.*

$$\operatorname{tr}\Big|_{V_0} o(a[-1]b[-1]c[-1]d) = \tag{28}$$

$$-\frac{1}{720}\big(4\langle [a,b]|[c,d]\rangle + 5\langle [a,d]|[b,c]\rangle\big) + \frac{1}{1152}\sum_{\pi\in S_4}\langle \pi(a)|\pi(b)\rangle\langle \pi(c)|\pi(d)\rangle$$

$$\operatorname{tr}\Big|_{V_1} o(a[-1]b[-1]c[-1]d) = \tag{29}$$

$$\begin{aligned}&-\frac{n+240}{720}\big(4\langle [a,b]|[c,d]\rangle + 5\langle [a,d]|[b,c]\rangle\big) + \frac{n-48}{1152}\sum_{\pi\in S_4}\langle \pi(a)|\pi(b)\rangle\langle \pi(c)|\pi(d)\rangle \\ &-\frac{1}{48}\sum_{\pi\in S_4}\langle \pi(a)|\pi(b)\rangle\kappa(\pi(c),\pi(d)) + \frac{1}{6}\operatorname{tr}\Big|_{V_1}\sum_{\pi\in S_3} a(0)\pi\big(b(0)\big)\pi\big(c(0)\big)\pi\big(d(0)\big)\end{aligned}$$

Proof. First, we need to expand $a[-1]b[-1]c[-1]d$ in terms of round brackets. We apply equation (9),

$$v[m] = \operatorname{Res}_z Y(v,z)(\log(1+z))^m(1+z)^{\operatorname{wt} v-1},$$

with $m=-1$, $\operatorname{wt} v = 1$ and inserting the expansion

$$(\log(1+z))^{-1} = z^{-1} + \frac{1}{2} - \frac{1}{12}z + \frac{1}{24}z^2 - \frac{19}{720}z^3\cdots.$$

We get

$$\begin{aligned}c[-1]d &= c(-1)d + \frac{1}{2}c(0)d - \frac{1}{12}c(1)d, \\ &= c(-1)d + \frac{1}{2}[c,d] - \frac{1}{12}\langle c|d\rangle\mathbf{1},\end{aligned}$$

$$\begin{aligned}
b[-1]c[-1]d &= b(-1)c(-1)d + \frac{1}{2}b(0)c(-1)d - \frac{1}{12}b(1)c(-1)d + \frac{1}{24}b(2)c(-1)d \\
&\quad + \frac{1}{2}b(-1)[c,d] + \frac{1}{4}b(0)[c,d] - \frac{1}{24}b(1)[c,d] - \frac{1}{12}\langle c|d\rangle b(-1)\mathbf{1}, \\
&= b(-1)c(-1)d + \frac{1}{2}b(0)c(-1)d - \frac{1}{12}b(1)c(-1)d + \frac{1}{24}b(2)c(-1)d \\
&\quad + \frac{1}{2}b(-1)[c,d] + \frac{1}{4}[b,[c,d]] - \frac{1}{24}\langle b|[c,d]\rangle\mathbf{1} - \frac{1}{12}\langle c|d\rangle b,
\end{aligned}$$

and

$$\begin{aligned}
a[-1]b[-1]c[-1]d &= a(-1)b(-1)c(-1)d + \tfrac{1}{2}a(0)b(-1)c(-1)d - \tfrac{1}{12}a(1)b(-1)c(-1)d \\
&+ \tfrac{1}{24}a(2)b(-1)c(-1)d - \tfrac{19}{720}a(3)b(-1)c(-1)d + \tfrac{1}{2}a(-1)b(0)c(-1)d + \tfrac{1}{4}a(0)b(0)c(-1)d \\
&- \tfrac{1}{24}a(1)b(0)c(-1)d + \tfrac{1}{48}a(2)b(0)c(-1)d - \tfrac{1}{12}a(-1)b(1)c(-1)d - \tfrac{1}{24}a(0)b(1)c(-1)d \\
&+ \tfrac{1}{144}a(1)b(1)c(-1)d + \tfrac{1}{24}a(-1)b(2)c(-1)d + \tfrac{1}{2}a(-1)b(-1)[c,d] + \tfrac{1}{4}a(0)b(-1)[c,d] \\
&- \tfrac{1}{24}a(1)b(-1)[c,d] + \tfrac{1}{48}a(2)b(-1)[c,d] + \tfrac{1}{4}a(-1)[b,[c,d]] + \tfrac{1}{8}[a,[b,[c,d]]] \\
&- \tfrac{1}{48}\langle a|[b,[c,d]]\rangle\mathbf{1} - \tfrac{1}{24}\langle b|[c,d]\rangle a - \tfrac{1}{12}\langle c|d\rangle a(-1)b - \tfrac{1}{24}\langle c|d\rangle [a,b] + \tfrac{1}{144}\langle c|d\rangle\langle a|b\rangle\mathbf{1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
o(a[-1]b[-1]c[-1]d) &= (a(-1)b(-1)c(-1)d)(3) + \tfrac{1}{2}\Big(a(0)b(-1)c(-1)d + a(-1)b(0)c(-1)d \\
&+ a(-1)b(-1)[c,d]\Big)(2) - \tfrac{1}{12}\Big(a(1)b(-1)c(-1)d + a(-1)b(1)c(-1)d + \langle c|d\rangle a(-1)b\Big)(1) \\
&+ \tfrac{1}{4}\Big(a(0)b(0)c(-1)d + a(0)b(-1)[c,d] + a(-1)[b,[c,d]]\Big)(1) + \tfrac{1}{24}\Big(a(2)b(-1)c(-1)d \\
&+ a(-1)b(2)c(-1)d - a(1)b(0)c(-1)d - a(0)b(1)c(-1)d - a(1)b(-1)[c,d] - \langle b|[c,d]\rangle a \\
&- \langle c|d\rangle [a,b] + 3[a,[b,[c,d]]]\Big)(0) - \tfrac{19}{720}(a(3)b(-1)c(-1)d)(-1) + \tfrac{1}{48}\Big(a(2)b(0)c(-1)d \\
&+ a(2)b(-1)[c,d] - \langle [a,b]|[c,d]\rangle\mathbf{1}\Big)(-1) + \tfrac{1}{144}\Big(a(1)b(1)c(-1)d + \langle c|d\rangle\langle a|b\rangle\mathbf{1}\Big)(-1).
\end{aligned} \tag{30}$$

If v is a weight-zero vector, then its zero mode $o(v)$ is $v(-1)$. Thus we refer to the zero-modes of weight-zero vectors in the above expansion $o(a[-1]b[-1]c[-1]d)$ as the -1-terms, likewise the zero modes of weight-one vectors are the 0-terms, those of the weight-two vectors are the 1-terms, etc.

First we will find the contribution of the -1-terms to the traces of $o(a[-1]b[-1]c[-1]d)$. Using the affine-Lie-algebra commutation relations, both of the -1-terms $a(3)b(-1)c(-1)d$ and

$a(2)b(0)c(-1)d$ equal $[a,b](2)c(-1)d$, and

$$\begin{aligned}[a,b](2)c(-1)d &= [[a,b],c](1)d \\ &= \langle [[a,b],c] | d\rangle \mathbf{1} \\ &= \langle [a,b] | [c,d]\rangle \mathbf{1}.\end{aligned}$$

Additionally, $a(2)b(-1)[c,d] = [a,b](1)[c,d] = \langle [a,b]|[c,d]\rangle\mathbf{1}$. Thus the terms $a(3)b(-1)c(-1)d$, $a(2)b(0)c(-1)d$ and $a(2)b(-1)[c,d]$ all equal $\langle [a,b]|[c,d]\rangle\mathbf{1}$. The term $a(1)b(1)c(-1)d$ is slightly more complicated, but still easy.

$$\begin{aligned}a(1)b(1)c(-1)d &= a(1)[b(1),c(-1)]d + a(1)c(-1)b(1)d \\ &= a(1)[b,c](0)d + a(1)\langle b|c\rangle d + a(1)c(-1)\langle b|d\rangle \mathbf{1} \\ &= \langle a|[[b,c],d]\rangle \mathbf{1} + \langle a|d\rangle\langle b|c\rangle \mathbf{1} + \langle a|c\rangle\langle b|d\rangle \mathbf{1} \\ &= -\langle [a,d]|[b,c]\rangle \mathbf{1} + \langle a|d\rangle\langle b|c\rangle \mathbf{1} + \langle a|c\rangle\langle b|d\rangle \mathbf{1}\end{aligned}$$

So referring to equation (30), we rewrite the the -1-terms as

$$\Big(-\frac{1}{180}\langle [a,b]|[c,d]\rangle - \frac{1}{144}\langle [a,d]|[b,c]\rangle + \frac{1}{144}\big(\langle a|b\rangle\langle c|d\rangle + \langle a|c\rangle\langle b|d\rangle + \langle a|d\rangle\langle b|c\rangle\big)\Big)\mathbf{1}(-1),$$

which equals

$$\Big(-\frac{1}{720}\big(4\langle [a,b]|[c,d]\rangle + 5\langle [a,d]|[b,c]\rangle\big) + \frac{1}{1152}\sum_{\pi\in S_4}\langle \pi(a)|\pi(c)\rangle\langle \pi(b)|\pi(d)\rangle\Big)\,\mathrm{id}\,.$$

From the creation axiom, only the -1-terms of $o(a[-1]b[-1]c[-1]d)$ contribute to its trace on $V_0 = \mathbb{C}\mathbf{1}$. So the trace of $o(a[-1]b[-1]c[-1]d)$ on V_0 follows immediately from the above equation. The same equation implies that the -1-terms contribute

$$-\frac{n}{720}\big(4\langle [a,b]|[c,d]\rangle + 5\langle [a,d]|[b,c]\rangle\big) + \frac{n}{1152}\sum_{\pi\in S_4}\langle \pi(a)|\pi(c)\rangle\langle \pi(b)|\pi(d)\rangle \tag{31}$$

to the trace of $o(a[-1]b[-1]c[-1]d)$ on V_1. Recall that $n = \dim V_1$.

Next we turn to the 1-terms,

$$-\frac{1}{12}\Big(a(1)b(-1)c(-1)d + a(-1)b(1)c(-1)d + \langle c|d\rangle a(-1)b\Big)(1) \\ + \frac{1}{4}\Big(a(0)b(0)c(-1)d + a(0)b(-1)[c,d] + a(-1)[b,[c,d]]\Big)(1).$$

Expand $a(-1)b(1)c(-1)d$ and $a(1)b(-1)c(-1)d$:

$$\begin{aligned} a(-1)b(1)c(-1)d &= a(-1)[b(1),c(-1)]d + a(-1)c(-1)b(1)d, \\ &= a(-1)[b,c](0)d + a(-1)\langle b|c\rangle d + a(-1)c(-1)\langle b|d\rangle \mathbf{1}, \\ &= a(-1)[[b,c],d] + \langle b|c\rangle a(-1)d + \langle b|d\rangle a(-1)c, \end{aligned}$$

$$\begin{aligned} a(1)b(-1)c(-1)d &= [a,b](0)c(-1)d + \langle a|b\rangle c(-1)d + b(-1)a(1)c(-1)d, \\ &= [[a,b],c](-1)d + c(-1)[[a,b],d] + \langle a|b\rangle c(-1)d \\ &\quad + b(-1)[[a,c],d] + \langle a|c\rangle b(-1)d + \langle a|d\rangle b(-1)c. \end{aligned}$$

Denote the trace of $(u(-1)v)(1)$ on V_1 by $(u:v)$. Notice that because this trace is a linear combination of symmetric invariant bilinear forms (see Lemma 5.2, equation (22)), $(u:v)$ is itself a symmetric invariant bilinear form. Using the above expansion of $a(-1)b(1)c(-1)d$ we compute

$$\begin{aligned} \operatorname{tr}\Big|_{V_1}(a(-1)b(1)c(-1)d)(1) &= (a:[[b,c],d]) + \langle b|c\rangle(a:d) + \langle b|d\rangle(a:c) \\ &= -([a,d]:[b,c]) + \langle b|c\rangle(a:d) + \langle b|d\rangle(a:c). \end{aligned}$$

We use the expansion $(a(1)b(-1)c(-1)d)(1)$ and identity (20) to compute

$$\begin{aligned} \operatorname{tr}\Big|_{V_1}(a(1)b(-1)c(-1)d)(1) &= ([[a,b],c]:d) + (c:[[a,b],d]) + \langle a|b\rangle(c:d) \\ &\quad + (b:[[a,c],d]) + \langle a|c\rangle(b:d) + \langle a|d\rangle(b:c), \end{aligned}$$

$$\begin{aligned}\operatorname{tr}\Big|_{V_1}(a(1)b(-1)c(-1)d)(1) &= ([a,b]:[c,d]) - ([a,b]:[c,d]) - ([a,c]:[b,d]) \\ &\quad + \langle a|b\rangle(c:d) + \langle a|c\rangle(b:d) + \langle a|d\rangle(b:c), \\ &= -([a,b]:[c,d]) - ([a,d]:[b,c]) + \langle a|b\rangle(c:d) \\ &\quad + \langle a|c\rangle(b:d) + \langle a|d\rangle(b:c).\end{aligned}$$

Lemma 5.2, equation (23) tells us that the trace of $o(a(0)b(-1)[c,d])$ on V_1 is zero. Additionally, $a(0)b(0)c(-1)d = [a,b](0)c(-1)d$, so it also has zero trace on V_1.

We sum to compute the contribution of the 1-terms to the trace of $o(a[-1]b[-1]c[-1]d)$ on V_1,

$$\begin{aligned}&\frac{1}{3}([a,b]:[c,d]) + \frac{1}{6}([a,d]:[b,c]) - \frac{1}{12}\Big(\langle a|b\rangle(c:d) + \langle a|c\rangle(b:d) \\ &\qquad + \langle a|d\rangle(b:c) + \langle b|c\rangle(a:d) + \langle b|d\rangle(a:c) + \langle c|d\rangle(a:b)\Big).\end{aligned}$$

Now substitute $(u:v) = \operatorname{tr}\big|_{V_1}(u(-1)v)(1) = 2\langle u|v\rangle + \kappa(u,v)$, equation (22), to get

$$\begin{aligned}&\frac{2}{3}\langle [a,b]|[c,d]\rangle + \frac{1}{3}\kappa([a,b],[c,d]) + \frac{1}{3}\langle [a,d]|[b,c]\rangle + \frac{1}{6}\kappa([a,d],[b,c]) - \frac{1}{3}\Big(\langle a|b\rangle\langle c|d\rangle + \langle a|c\rangle\langle b|d\rangle \\ &+\langle a|d\rangle\langle b|c\rangle\Big) - \frac{1}{12}\Big(\langle a|b\rangle\kappa(c,d) + \langle a|c\rangle\kappa(b,d) + \langle a|d\rangle\kappa(b,d) + \langle b|c\rangle\kappa(a,d) + \langle b|d\rangle\kappa(a,c) + \langle c|d\rangle\kappa(a,b)\Big),\end{aligned}$$

which equals

$$\begin{aligned}&\frac{2}{3}\langle [a,b]|[c,d]\rangle + \frac{1}{3}\langle [a,d]|[b,c]\rangle + \frac{1}{3}\kappa([a,b],[c,d]) + \frac{1}{6}\kappa([a,d],[b,c]) \\ &\qquad - \frac{1}{24}\sum_{\pi\in S_4}\langle \pi(a)|\pi(b)\rangle\langle \pi(c)|\pi(d)\rangle - \frac{1}{48}\sum_{\pi\in S_4}\langle \pi(a)|\pi(b)\rangle\kappa(\pi(c),\pi(d)). \qquad (32)\end{aligned}$$

Now let's do the 2-terms. From equation (30), these are

$$\frac{1}{2}\Big(a(0)b(-1)c(-1)d + a(-1)b(0)c(-1)d + a(-1)b(-1)[c,d]\Big)(2).$$

Expand the first two terms:

$$\begin{aligned} a(0)b(-1)c(-1)d &= [a,b](-1)c(-1)d + b(-1)a(0)c(-1)d, \\ &= [a,b](-1)c(-1)d + b(-1)[a,c](-1)d + b(-1)c(-1)[a,d], \\ a(-1)b(0)c(-1)d &= a(-1)[b,c](-1)d + a(-1)c(-1)[b,d]. \end{aligned}$$

Using Lemma 5.2, equation (24), we compute the contribution of the 2-terms to the trace,

$$\begin{aligned} \frac{1}{2}\Big(&- \langle [[a,b],c] | d\rangle - \langle [b,[a,c]] | d\rangle - \langle [b,c] | [a,d]\rangle - \langle [a,[b,c]] | d\rangle - \langle [a,c] | [b,d]\rangle \\ &- \langle [a,b] | [c,d]\rangle + \operatorname{tr}\Big|_{V_1} \big(d(0)c(0)[a,b](0) + d(0)[a,c](0)b(0) + [a,d](0)c(0)b(0) \\ &+ d(0)[b,c](0)a(0) + [b,d](0)c(0)a(0) + [c,d](0)b(0)a(0)\big)\Big). \end{aligned}$$

To simplify this expression we use the properties of $\langle\ |\ \rangle$ including equation (20) on the first six terms; for the remaining terms, we expand the commutators using $[u,v](0) = u(0)v(0) - v(0)u(0)$ and then use $\operatorname{tr} AB = \operatorname{tr} BA$ to move $a(0)$ to the left of each term. We find that the contribution of the 2-terms to the trace of $o(a[-1]b[-1]c[-1]d)$ on V_1 is

$$-\langle [a,b] | [c,d]\rangle + \frac{1}{2}\operatorname{tr}\Big|_{V_1} \big(a(0)b(0)d(0)c(0) + a(0)c(0)d(0)b(0) - 2a(0)d(0)c(0)b(0)\big). \qquad (33)$$

From Lemma 5.2, equation (25), the contribution of the 3-term, $(a(-1)b(-1)c(-1)d)(3)$, to the trace of $o(a[-1]b[-1]c[-1]d)$ on V_1 is

$$-\langle [a,b] | [c,d]\rangle - 2\langle [a,d] | [b,c]\rangle + \operatorname{tr}\Big|_{V_1} a(0)d(0)c(0)b(0). \qquad (34)$$

Lemma 5.2, equation (21) implies that the 0-terms have no contribution to the trace on V_1. We can now add the contributions of the -1, 1, 2 and 3-terms from expressions (31), (32), (33) and (34) to find that

$$\begin{aligned}\operatorname{tr}\Big|_{V_1} o(a[-1]b[-1]c[-1]d) = &-\frac{n+240}{720}\Big(4\langle[a,b]|[c,d]\rangle - 5\langle[a,d]|[b,c]\rangle\Big) + \frac{1}{3}\kappa([a,b],[c,d]) \\ &+\frac{1}{6}\kappa([a,d],[b,c]) + \frac{1}{2}\operatorname{tr}\Big|_{V_1}\big(a(0)b(0)d(0)c(0) + a(0)c(0)d(0)b(0)\big) \\ &+\frac{n-48}{1152}\textstyle\sum_{\pi\in S_4}\langle\pi(a)|\pi(b)\rangle\langle\pi(c)|\pi(d)\rangle - \frac{1}{48}\textstyle\sum_{\pi\in S_4}\langle\pi(a)|\pi(b)\rangle\kappa(\pi(c),\pi(d))\end{aligned} \tag{35}$$

We can uncover another symmetry in this trace by expanding $\kappa([a,b],[c,d])$ and $\kappa([a,d],[b,c])$.

$$\begin{aligned}\kappa([a,b],[c,d]) &= \operatorname{tr}\Big|_{V_1}[a,b](0)[c,d](0) \\ &= \operatorname{tr}\Big|_{V_1}\big(a(0)b(0)c(0)d(0) - a(0)b(0)d(0)c(0) - a(0)c(0)d(0)b(0) + a(0)d(0)c(0)b(0)\big)\end{aligned}$$

Likewise,

$$\kappa([a,d],[b,c]) = \operatorname{tr}\Big|_{V_1}\big(a(0)d(0)b(0)c(0) - a(0)d(0)c(0)b(0) - a(0)b(0)c(0)d(0) + a(0)c(0)b(0)d(0)\big).$$

So,

$$\begin{aligned}\frac{1}{3}\kappa([a,b],[c,d]) + \frac{1}{6}\kappa([a,d],[b,c]) + \frac{1}{2}\operatorname{tr}\Big|_{V_1}\big(a(0)b(0)d(0)c(0) + a(0)c(0)d(0)b(0)\big) \\ = \frac{1}{6}\sum_{\pi\in S_3} a(0)\pi\big(b(0)\big)\pi\big(c(0)\big)\pi\big(d(0)\big).\end{aligned}$$

Substituting this into equation (35) completes the proof. □

6 Trace of $\upsilon(u,v)$

Having found the traces on V_0 and V_1 of the most general square-bracket-weight four elements of V that we can access using the affine Lie algebra $A(V_{[1]})$, we want to find linear combinations of these elements such that the trace on V_0 is zero and the trace on V_1 is non-zero.

Proposition 6.1. *Let V be a strongly rational VOA. If u and v are elements of V_1 such that*

$[u,v]=0$ *and* $\langle u|v\rangle = 0$*, and* k *is defined by* $\langle u|u\rangle = k\langle v|v\rangle$*, then*

$$\begin{aligned}
&\mathrm{tr}\Big|_{V_0} o(u[-1]^3u - 6ku[-1]^2v[-1]v + k^2v[-1]^3v) = 0,\\
&\mathrm{tr}\Big|_{V_1} o(u[-1]^3u - 6ku[-1]^2v[-1]v + k^2v[-1]^3v) = \mathrm{tr}\Big|_{V_1}\Big(u(0)^4 - 6ku(0)^2v(0)^2 + k^2v(0)^4\Big).
\end{aligned}$$

Proof. The proof is a straight forward application of Proposition 5.3, which implies that

$$\begin{aligned}
\mathrm{tr}\Big|_{V_0} o(u[-1]^3u) &= -\frac{1}{720}\big(4\langle [u,u]|[u,u]\rangle + 5\langle [u,u]|[u,u]\rangle\big) + \frac{24}{1152}\langle u|u\rangle^2,\\
&= \frac{1}{48}\langle u|u\rangle^2,\\
\mathrm{tr}\Big|_{V_0} o(u[-1]^2v[-1]v) &= -\frac{1}{720}\big(4\langle [u,u]|[v,v]\rangle + 5\langle [u,v]|[u,v]\rangle\big) + \frac{1}{1152}\big(16\langle u|v\rangle^2 + 8\langle u|u\rangle\langle v|v\rangle\big),\\
&= \frac{1}{144}\langle u|u\rangle\langle v|v\rangle.
\end{aligned}$$

Combining these we see that

$$\begin{aligned}
\mathrm{tr}\Big|_{V_0} o(u[-1]^3u - 6ku[-1]^2v[-1]v + k^2v[-1]^4) &= \frac{1}{48}\langle u|u\rangle^2 - 6k\frac{1}{144}\langle u|u\rangle\langle v|v\rangle + k^2\frac{1}{48}\langle v|v\rangle^2,\\
&= \frac{1}{48}\big(\langle u|u\rangle - k\langle v|v\rangle\big)^2,\\
&= 0.
\end{aligned}$$

Also from Proposition (5.3),

$$\mathrm{tr}\Big|_{V_1} o(u[-1]^3u) = \frac{n-48}{48}\langle u|u\rangle^2 - \frac{1}{2}\langle u|u\rangle\kappa(u,u) + \mathrm{tr}\Big|_{V_1} u(0)^4.$$

For $\mathrm{tr}\big|_{V_1} o(u[-1]v[-1]u[-1]v)$, we note that $u(0)$ commutes with $v(0)$ by assumption, so

$$\mathrm{tr}\big|_{V_1} u(0)\pi\big(u(0)\big)\pi\big(v(0)\big)\pi\big(v(0)\big) = \mathrm{tr}\big|_{V_1} u(0)^2v(0)^2$$

for all $\pi \in S_3$. We also note that

$$\sum_{\pi \in S_4} \langle \pi(u)|\pi(u)\rangle \kappa(\pi(v), \pi(v)) = 4\langle u|u\rangle \kappa(v,v) + 4\langle v|v\rangle \kappa(u,u) + 16\langle u|v\rangle \kappa(u,v),$$

$$= 4\langle u|u\rangle \kappa(v,v) + 4\langle v|v\rangle \kappa(u,u).$$

So we have

$$\operatorname{tr}\Big|_{V_1} o(u[-1]^2 v[-1]v) = \frac{n-48}{144}\langle u|u\rangle\langle v|v\rangle - \frac{1}{12}\langle u|u\rangle \kappa(v,v) - \frac{1}{12}\langle v|v\rangle \kappa(u,u) + \operatorname{tr}\Big|_{V_1} u(0)^2 v(0)^2.$$

Thus

$$\begin{aligned}
\operatorname{tr}\Big|_{V_1} o(u[-1]^3 u - 6ku[-1]^2 v[-1]v + k^2 v[-1]^4) &= \frac{n-48}{48}\langle u|u\rangle^2 - \frac{1}{2}\langle u|u\rangle \kappa(u,u) + \operatorname{tr}\Big|_{V_1} u(0)^4 \\
&\quad - 6k\Big(\frac{n-48}{144}\langle u|u\rangle\langle v|v\rangle - \frac{1}{12}\langle u|u\rangle\kappa(v,v) - \frac{1}{12}\langle v|v\rangle \kappa(u,u) + \operatorname{tr}\Big|_{V_1} u(0)^2 v(0)^2\Big) \\
&\quad + k^2\Big(\frac{n-48}{48}\langle v|v\rangle^2 - \frac{1}{2}\langle v|v\rangle\kappa(v,v) + \operatorname{tr}\Big|_{V_1} v(0)^4\Big), \\
&= \frac{n-48}{48}\big(\langle u|u\rangle - k\langle v|v\rangle\big)^2 - \frac{\kappa(u,u)}{2}\big(\langle u|u\rangle - k\langle v|v\rangle\big) + \frac{k\kappa(v,v)}{2}\big(\langle u|u\rangle - k\langle v|v\rangle\big) \\
&\quad + \operatorname{tr}\Big|_{V_1}\Big(v(0)^4 - 6ku(0)^2 v(0)^2 + k^2 v(0)^4\Big), \\
&= \operatorname{tr}\Big|_{V_1}\Big(v(0)^4 - 6ku(0)^2 v(0)^2 + k^2 v(0)^4\Big).
\end{aligned}$$

□

Assume that V_1 is a semisimple Lie algebra with decomposition

$$V_1 = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m$$

into simple Lie algebras such that $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ for $i \neq j$. Let π_i be the projection into $\mathfrak{g}_i$ with respect to this decomposition.

Lemma 6.2. *Let $\langle\ |\ \rangle$ be an invariant bilinear form on V_1. Then for all $u, v \in \mathfrak{g}$,*

$$\langle u|v\rangle = \sum_{i=1}^{m} \lambda_i \kappa(\pi_i(u), \pi_i(v))$$

for some $\lambda_i \in \mathbb{C}$.

Proof. Clearly, $\langle u|v\rangle = \sum_{i,j=1}^{m} \langle \pi_i(u)|\pi_j(v)\rangle$. Because $\mathfrak{g}_j$ is simple and non-abelian, $\mathfrak{g}_j = [\mathfrak{g}_j, \mathfrak{g}_j]$, and there exist x and y in $\mathfrak{g}_j$ such that $\pi_j(v) = [x, y]$. Assume $j \neq i$. We compute

$$\langle \pi_i(u)|\pi_j(v)\rangle = \langle \pi_i(u)|[x, y]\rangle = \langle [\pi_i(u), x]|y\rangle = \langle 0|y\rangle = 0.$$

Thus $\langle \mathfrak{g}_i|\mathfrak{g}_j\rangle = 0$ for $i \neq j$, and $\langle u|v\rangle = \sum_{i=1}^{m} \langle \pi_i(u)|\pi_i(v)\rangle$.

Fix an i, and consider $\mathfrak{g}_i$ as a $\mathfrak{g}_i$-module under the adjoint action and $\mathfrak{g}_i^*$ as a $\mathfrak{g}_i$-module via $g \cdot f(v) = f([v, g])$. Any invariant bilinear form induces a $\mathfrak{g}_i$-module morphism from $\mathfrak{g}_i$ to $\mathfrak{g}_i^*$. Call the morphism induced from $\langle\ |\ \rangle$, M, and the one induced from the Killing from, K. Since the Killing form is non-degenerate on $\mathfrak{g}_i$ and $\mathfrak{g}_i$ is finite-dimensional, K is an isomorphism. Thus $K^{-1} \circ M$ is a $\mathfrak{g}_i$-module endomorphism of $\mathfrak{g}_i$, which is irreducible. By Shur's lemma, $K^{-1} \circ M = \lambda_i \,\mathrm{id}$ for some $\lambda_i \in \mathbb{C}$. So on each $\mathfrak{g}_i$, $\langle\ |\ \rangle = \lambda_i \kappa(\ ,\ )$. ☐

Proposition 6.1 requires that $[u, v] = 0$ and $\langle u|v\rangle = 0$. By choosing u and v from the Cartan subalgebra of V_1, we can fulfill the condition $[u, v] = 0$. From Lemma 6.2, in order to make $\langle u|v\rangle = 0$, we can either choose u and v in different simple components of V_1 or pick them in the same component such that $\kappa(u, v) = 0$.

For any u in $\mathfrak{h}$, define $u^* := \kappa(u,\ )$ to be its dual in $\mathfrak{h}^*$.

Lemma 6.3. *Let u and v be elements of $\mathfrak{h}$, the Cartan subalgebra of V_1. Then*

$$\mathrm{tr}\Big|_{V_1} u(0)^4 = \sum_{\alpha \in \Phi} (u^*, \alpha)^4,$$

$$\mathrm{tr}\Big|_{V_1} u(0)^2 v(0)^2 = \sum_{\alpha \in \Phi} (u^*, \alpha)^2 (v^*, \alpha)^2,$$

where $(\ , \ )$ is the form induced on $\mathfrak{h}^*$ by the Killing form and Φ is the root system of V_1.

Proof. Let V_1 have Cartan decomposition $\mathfrak{h} \oplus \sum_{\alpha\in\Phi} L_\alpha$. Each L_α is one dimensional and $[h,a] = \alpha(h)a$ for $a \in L_\alpha$ and $h \in \mathfrak{h}$. So it is clear that

$$\operatorname{tr}\Big|_{V_1} u(0)^4 = \sum_{\alpha\in\Phi} \alpha(u)^4,$$
$$\operatorname{tr}\Big|_{V_1} u(0)^2 v(0)^2 = \sum_{\alpha\in\Phi} \alpha(u)^2\alpha(v)^2.$$

Denoting the inverse of the $u \mapsto u^*$ map by $\alpha \mapsto t_\alpha$, $\alpha(u) = \kappa(t_\alpha, u) = (\alpha, u^*)$. The results follow. □

Let u and v be orthonormal elements of the Cartan subalgebra of V_1, and define

$$\upsilon(u,v) := o(u[-1]^3u - 6u[-1]^2v[-1]v + v[-1]^3v). \tag{36}$$

Proposition 6.1 and Lemma 6.3 imply that

$$\operatorname{tr}\Big|_{V_1} \upsilon(u,v) = \sum_{\alpha\in\Phi_{V_1}} (u^*,\alpha)^4 - 6(u^*,\alpha)^2(v^*,\alpha)^2 + (v^*,\alpha)^4. \tag{37}$$

Notice that the restriction on the lengths of u and v is just for convenience. Indeed, if $\langle u|u\rangle = k\langle v|v\rangle \neq 0$, then

$$\operatorname{tr}\big|_{V_1} o(u[-1]^3u - 6ku[-1]^2v[-1]v + k^2v[-1]^3v) = k^2 \operatorname{tr}\big|_{V_1} \upsilon(u',v),$$

where $u = \sqrt{k}u'$. So finding perpendicular u and v such that $\operatorname{tr}\big|_{V_1} o(u[-1]^3u - 6ku[-1]^2v[-1]v + k^2v[-1]^3v) \neq 0$ amounts to finding orthonormal u and v so that $\operatorname{tr}\big|_{V_1} \upsilon(u,v) \neq 0$.

Lemma 6.4. *Let V be a strongly rational VOA and let V_1 have a Lie algebra decomposition:*

$$V_1 = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m \oplus \mathfrak{h},$$

such that each $\mathfrak{g}_i$ is simple and $\mathfrak{h}$ is abelian. If $m > 1$, then there exist u and v such that $\operatorname{tr}\big|_{V_1} \upsilon(u,v) \neq 0$.

Proof. Choose u and v to be unit vectors in the Cartan subalgebras of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. Let $\Phi = \Phi_1 + \Phi_2 + \cdots + \Phi_m$ be the root-system decomposition corresponding to our Lie-algebra decomposition. Because Φ_i and Φ_j are orthogonal, $(u^*, \alpha) = 0$ for $\alpha \in \Phi_i$ with $i \neq 1$. Likewise, $(v^*, \alpha) = 0$ for $\alpha \in \Phi_i$ with $i \neq 2$. Thus from equation (37),

$$\operatorname{tr}\Big|_{V_1} \upsilon(u,v) = \sum_{\alpha\in\Phi} \Big((u^*,\alpha)^4 - 6(u^*,\alpha)^2(v^*,\alpha)^2 + (v^*,\alpha)^4\Big) = \sum_{\alpha\in\Phi_1} (u^*,\alpha)^4 + \sum_{\alpha\in\Phi_2} (v^*,\alpha)^4.$$

Because $(\ ,\ )$ is non-degenerate on $\mathfrak{h}_1 = \operatorname{Span}_{\mathbb{R}} \Phi_1$, there exists at least one $\beta \in \Phi_1$ such that $(u^*, \beta) \neq 0$. Furthermore, $(\ ,\ ) \in \mathbb{R}$, so

$$\operatorname{tr}\Big|_{V_1} \upsilon(u,v) = \sum_{\alpha\in\Phi_1} (u^*,\alpha)^4 + \sum_{\alpha\in\Phi_2} (v^*,\alpha)^4 \geq (u^*,\beta)^4 > 0.$$

□

Lemma 6.5. *Let V be a strongly rational VOA with the above decomposition of V_1. If $m = 1$ and $\mathfrak{g}_1$ is a simple Lie algebra of type A, B, C or D, but not A_1, A_2 or D_4, then there exist $u, v \in V_1$ such that* $\operatorname{tr}\big|_{V_1} \upsilon(u,v) \neq 0$.

Proof. We use the root systems listed in Appendix A and compute. For A_ℓ, $\ell \geq 3$, choose u so that $u^* = (e_1 - e_2)/\sqrt{2}$ and v so that $v^* = (e_\ell - e_{\ell+1})/\sqrt{2}$. Since $\ell \geq 3$, u^* and v^* are perpendicular. Using equation (37),

$$\begin{aligned}\operatorname{tr}\Big|_{A_\ell(\mathbb{C})} \upsilon(u,v) &= \frac{1}{4} \sum_{\alpha\in\Phi_{A_\ell}} (e_1 - e_2, \alpha)^4 - 6(e_1 - e_2, \alpha)^2(e_\ell - e_{\ell+1}, \alpha)^2 + (e_\ell - e_{\ell+1}, \alpha)^4, \\ &= \frac{1}{4}\big(2(\pm 2)^4 + 4(\ell-1)(\pm 1)^4 - 6\cdot 8(\pm 1)^2(\pm 1)^2 + 2(\pm 2)^4 + 4(\ell-1)(\pm 1)^4\big), \\ &= 2\ell + 2.\end{aligned}$$

So for $\ell \geq 3$, $\operatorname{tr}\big|_{A_\ell(\mathbb{C})} \upsilon(u,v) \neq 0$. For B_ℓ, C_ℓ and D_ℓ, we choose u so that $u^* = e_1$ and v so that $v^* = e_\ell$. From equation (37),

$$\begin{aligned}
\operatorname{tr}\Big|_{B_\ell(\mathbb{C})} \upsilon(u,v) &= \sum_{\alpha\in\Phi_{B_\ell}} (e_1,\alpha)^4 - 6(e_1,\alpha)^2(e_\ell,\alpha)^2 + (e_\ell,\alpha)^4,\\
&= 2(\pm1)^4 + 4(\ell-1)(\pm1)^4 - 6\cdot 4(\pm1)^2(\pm1)^2 + 2(\pm1)^4 + 4(\ell-1)(\pm1)^4,\\
&= 8(\ell-4)+4;
\end{aligned}$$

$$\begin{aligned}
\operatorname{tr}\Big|_{C_\ell(\mathbb{C})} \upsilon(u,v) &= 2(\pm2)^4 + 4(\ell-1)(\pm1)^4 - 6\cdot 4(\pm1)^2(\pm1)^2 + 2(\pm2)^4 + 4(\ell-1)(\pm1)^4,\\
&= 8(\ell+4);
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{tr}\Big|_{D_\ell(\mathbb{C})} \upsilon(u,v) &= 4(\ell-1)(\pm1)^4 - 6\cdot 4(\pm1)^2(\pm1)^2 + 4(\ell-1)(\pm1)^4,\\
&= 8\ell - 32.
\end{aligned}$$

Therefore $\operatorname{tr}\big|_{B_\ell(\mathbb{C})}\upsilon(u,v) \neq 0$ for any integer ℓ, $\operatorname{tr}\big|_{C_\ell(\mathbb{C})}\upsilon(u,v) \neq 0$ for any positive ℓ and $\operatorname{tr}\big|_{D_\ell(\mathbb{C})}\upsilon(u,v) \neq 0$ for $\ell \neq 4$. $\square$

Proposition 6.6. *Let V be a strongly holomorphic $c = 24$ VOA, such that V_1 is semisimple. Unless V_1 is an exceptional Lie algebra or of type A_1, A_2 or D_4, there exist u and v orthonormal elements of the Cartan subalgebra of V_1 such that some multiple of $\upsilon(u,v)$ has graded trace equal to $E_4(\tau)$.*

Proof. Because V is of strong CFT-type, $V_{[1]} = V_1$. This implies that $\upsilon(u,v) \in V_{[4]}$, so its graded trace has weight four. Because V is strongly holomorphic and has central charge 24, Proposition 6.1 assures us that the graded trace of $\upsilon(u,v)$ is holomorphic, and Lemmas 6.4 and 6.5 give u and v so that the graded trace is non-zero, in all cases expect those noted. The result follows because the space of weight-four, holomorphic modular forms is $\mathbb{C}E_4(\tau)$. $\square$

For V_1 an exceptional Lie algebra or a Lie algebra of type A_1, A_2 or D_4, we show in

Section 7 that there are no elements u and v such that $v(u, v)$ has a non-zero graded trace. However, for those cases, in Section 8 we find other elements of $V_{[4]}$ with non-zero holomorphic graded traces.

7 Spherical harmonics

Let $\{x_1, x_2, \ldots, x_n\}$ be coordinate functions for some orthonormal basis of a vector space, W. A polynomial $P(x_1, x_2, \ldots, x_n)$ is called a *spherical harmonic*, if it satisfies the Laplace equation, that is, $\Delta P := \sum_{i=1}^{n} \frac{\partial^2 P}{\partial x_i^2} = 0$. Assume that P is a spherical harmonic, and let T be an element of the orthogonal group, $O(n)$. Define $(T \bullet P)(x_1, x_2, \ldots, x_n) := P(T^{-1}(x_1, x_2, \ldots, x_n))$. Then the polynomial $T \bullet P$ is also a spherical harmonic [ABR92].

More specifically, let W equal $\mathfrak{h}^*$, the dual of the Cartan subalgebra of V_1, and choose another orthonormal basis for $\mathfrak{h}^*$ with coordinate functions $\{y_1, y_2, \ldots, y_n\}$. Define

$$Q(y_1, y_2, \ldots, y_n) := y_1^4 - 6y_1^2 y_2^2 + y_2^4. \tag{38}$$

It is easy to check that Q is a spherical harmonic with respect to $\Delta_y = \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2}$. The change of basis matrix T, so that $T\mathbf{y} = \mathbf{x}$, is orthogonal. So

$$P := T \bullet Q = Q(T^{-1}(x_1, x_2, \ldots, x_n))$$

is a spherical harmonic with respect to $\Delta_x = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

In Section 6, we chose orthonormal vectors u and v in $\mathfrak{h}$, defined the element $v(u, v) \in V_{[4]}$, equation (36), and computed the trace of $o(v(u, v))$ on V_1. Now define our orthonormal basis of $\mathfrak{h}^*$, $\{u^*, v^*, \ldots\}$, so that y_1 is the coordinate function for u^* and y_2 is the coordinate function for v^*, and let $(a_1, a_2, \ldots, a_n)$ be the coordinate vector of α with respect to this basis. Then from equation (38),

$$Q(a_1, a_2, \ldots, a_n) = (u^*, \alpha)^4 - 6(u^*, \alpha)^2(v^*, \alpha)^2 + (v^*, \alpha)^4.$$

Let $(b_1, b_2, \ldots, b_n)$ be the coordinate vector of α with respect to our original orthonormal basis. Then

$$P(\alpha) := P(b_1, b_2, \ldots, b_n) = Q(T^{-1}(b_1, b_2, \ldots, b_n)) = Q(a_1, a_2, \ldots, a_n).$$

We can now rewrite equation (37) as

$$\operatorname{tr}\Big|_{V_1} o(v(u, v)) = \sum_{\alpha \in \Phi_{V_1}} P(\alpha), \tag{39}$$

for some spherical harmonic P. Since Q is homogeneous of degree four and T is linear, P is either homogeneous of degree four or identically zero. Because $T^{-1} \bullet Q = P$, P cannot be identically zero, so it is homogeneous of degree four.

At the end of Section 6, we claimed that for V_1 exceptional or of type A_1, A_2 or D_4, there is no choice of u and v such that the graded trace of $v(u, v)$ is non-zero, For strongly holomorphic $c = 24$ VOAs, $Z(v(u, v), \tau) = cE_e(\tau)$. so it suffices to prove that $\operatorname{tr}\big|_{V_1} o(u, v) = 0$ for all choices of u and v. By the above discussion, this follows from the following proposition.

Proposition 7.1. *Let* V_1 *be a simple Lie algebra of type* A_1*,* A_2*,* D_4*,* E*,* F *or* G*. Then* $\sum_{\alpha \in \Phi_{V_1}} P(\alpha) = 0$ *for all homogeneous degree-four spherical harmonics* P*.*

Proof. We can reduce the number of calculations needed to prove this by using the action of the Weyl group, W. Define

$$S := \frac{1}{|W|} \sum_{w \in W} w \bullet P.$$

Then S is invariant under the action of the Weyl group, and because $W \subset O(n)$, S is a spherical harmonic. We compute

$$\sum_{\alpha \in \Phi} S(\alpha) = \frac{1}{|W|} \sum_{w \in W} \sum_{\alpha \in \Phi} P(w^{-1}(\alpha)).$$

The Weyl group preserves the root system, so $\sum_{\alpha\in\Phi} P(w^{-1}(\alpha)) = \sum_{\alpha\in\Phi} P(\alpha)$, and

$$\sum_{\alpha\in\Phi} S(\alpha) = \sum_{\alpha\in\Phi} P(\alpha).$$

We have shown that if there exists a spherical harmonic P such that $\sum_{\alpha\in\Phi} P(\alpha) = c$. Then there exists a W-invariant spherical harmonic S such that $\sum_{\alpha\in\Phi} S(\alpha) = c$.

Assume that P is homogeneous of degree four. Then S is either homogeneous of degree four or identically zero. Thus to show that $\sum_{\alpha\in\Phi} P(\alpha) = 0$ for all homogeneous, degree-four spherical harmonics P, it suffices to show either that $\sum_{\alpha\in\Phi} S(\alpha) = 0$ for all homogeneous, degree-four, W-invariant spherical harmonics S, or that there are no such S.

We need to find the spaces of invariant spherical harmonics for root systems of type A_1, A_2, D_4, E, F and G. The space of W-invariant polynomials for a Weyl group W are well-known, see [Hum90, ch. 3]. It is a finitely-generated polynomial ring; call it R. For each root system, the degrees of the homogeneous generating polynomials of R are unique.

We are interested in the (W-)invariant polynomials of (homogeneous) degree four. Because there are no invariant polynomials of degree one, the space of degree-four invariants is spanned by the degree-four generators and products of two degree-two generators. For all root systems, there is exactly one degree-two generator of R, which we can take to be $x_1^2+x_2^2+\cdots+x_n^2$. Of the root systems we are considering, only type D_4 has any degree-four generators of R.

Thus for root systems of type A_1, A_2, E, F and G, the space degree-four invariants is spanned by $(x_1^2 + x_2^2 + \cdots + x_n^2)^2$. Clearly, this polynomial is not a spherical harmonic. Thus there are no W-invariant homogeneous degree-four spherical harmonics and $\sum_{\alpha\in\Phi} P(\alpha) = 0$ for all homogeneous degree-four spherical harmonics P.

For a root system of type D_4, there are two degree-four generators of R. We can take them to be $x_1^4+x_2^4+x_3^4+x_4^4$ and $x_1x_2x_3x_4$. Thus $\{(x_1^2+x_2^2+x_3+x_4^2)^2, x_1^4+x_2^4+x_3^4+x_4^4, x_1x_2x_3x_4\}$ is a basis for the space degree-four invariants. We compute the kernel of Δ on this space and get the following basis for the space of degree-four invariant spherical harmonics;

$$\{(x_1^2 + x_2^2 + x_3 + x_4^2)^2 - 2(x_1^4 + x_2^4 + x_3^4 + x_4^4), x_1x_2x_3x_4\}.$$

Denote $(x_1^2 + x_2^2 + x_3 + x_4^2)^2 - 2(x_1^4 + x_2^4 + x_3^4 + x_4^4)$ by p and $x_1x_2x_3x_4$ by q. Referring to Appendix A for the root system of type D_4, we compute

$$\sum_{\alpha \in \Phi_{D_4}} p(\alpha) = 24((1+1)^2 - 2(1+1)) = 0,$$
$$\sum_{\alpha \in \Phi_{D_4}} q(\alpha) = 0.$$

Therefore, $\sum_{\alpha \in \Phi_{D_4}} S(\alpha) = 0$ for all homogeneous degree-four W-invariant spherical harmonics S. So $\sum_{\alpha \in \Phi_{D_4}} P(\alpha) = 0$ for all homogeneous degree-four spherical harmonics P. □

7.1 Niemeier lattice VOAs

The choice of the vector $v(u, v)$ and the use of spherical harmonics are inspired by work of Dong, Mason and Nagatomo on lattice VOAs [DMN01]. Let L be a positive-definite even lattice of rank d. One can use L to define the lattice VOA, V_L, see [Don93]. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_d\}$ be an orthonormal basis of $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$, then $\alpha_{i_1}(-n_1)\alpha_{i_2}(-n_2) \cdots \alpha_{i_k}(-n_k)$, with $n_j > 0$, is an element of V_L with weight $\sum_{j=1}^k n_j$. Dong, Mason and Nagatomo proved that if $P(x_1, x_2, \ldots, x_d)$ is a spherical harmonic of degree k, then $v_P := P(\alpha_1(-1), \alpha_2(-1), \ldots, \alpha_d(-1))$ is a highest-weight vector of weight k, and

$$Z_{V_L}(v_P, \tau) = \frac{\Theta_L(P, \tau)}{\big(\eta(\tau)\big)^d}, \tag{40}$$

where $\Theta_L(P, \tau) = \sum_{\alpha \in L} P(\alpha) q^{\frac{(\alpha,\alpha)}{2}}$ and $\eta(\tau) = q^{1/24} \prod_{n>0}(1 - q^n)$ [DMN01, Theorem 3].

We can use this result to prove our main theorem for those lattice VOAs to which it applies, namely, for Niemeier lattice VOAs. To assure that V_L is holomorphic, we need L to be self-dual. To make the central charge of V_L equal 24, we need the rank of L to be 24. Even, self-dual, positive-definite lattices of rank 24 are called Niemeier lattices. They are classified by their square-length two vectors, which form root systems of A, D and E type (see Venkov's chapter in [CS99]). For the 23 Niemeier lattices with square-length two vectors, these are:

$$\begin{array}{ccccccc}
24A_1 & 12A_2 & 8A_3 & 6A_4 & 4A_6 & 3A_8 & 2A_{12} \\
A_{24} & 6D_4 & 4D_6 & 3D_8 & 2D_{12} & D_{24} & 4E_6 \\
3E_8 & 4A_5+D_4 & 2A_7+2D_5 & 2A_9+D_6 & A_{15}+D_9 & E_8+D_{16} & 2E_7+D_{10} \\
E_7+A_{17} & E_6+D_7+A_{11} & & & & &
\end{array}$$

There is also one rank-24, even, self-dual lattice with no square-length two vectors, the Leech lattice Λ. Our main theorem does not apply to the Leech lattice; so when we say Niemeier lattice, we exclude the Leech lattice.

We can apply Dong, Mason and Nagatomo's theorem to the Niemeier lattice case to get highest-weight vectors with graded trace $E_4(\tau)$. Since we want v_P to have weight 4, let P be a homogeneous degree-four spherical harmonic. Because L is even and positive-definite, only non-negative integral powers of q appear in the expansion of $\Theta_L(P,\tau)$, and because P is homogeneous of positive degree, $P(\alpha)=0$. Thus

$$\Theta_L(P,\tau) = \sum_{\substack{\alpha\in L \\ (\alpha,\alpha)=2}} P(\alpha)q + \sum_{\substack{\alpha\in L \\ (\alpha,\alpha)=4}} P(\alpha)q^2 + \cdots .$$

Since the rank of L is 24, from equation (40) we have

$$Z_{V_L}(v_P,\tau) = \sum_{\substack{\alpha\in L \\ (\alpha,\alpha)=2}} P(\alpha) + \cdots . \tag{41}$$

Because V_L is holomorphic, $Z_{V_L}(v_P,\tau) = cE_4(\tau)$ for some $c \in \mathbb{C}$. For each Niemeier lattice, we need to show that there exists P such that $c \neq 0$. From equation (41), it suffices to show that $\sum_{\substack{\alpha\in L \\ (\alpha,\alpha)=2}} P(\alpha) \neq 0$. Since, for each Niemeier lattice, $\{\alpha \in L | (\alpha,\alpha)=2\}$ is one of the above root systems, we need to show that there exists a homogeneous degree-four spherical harmonic P such that $\sum_{\alpha\in\Phi} P(\alpha) \neq 0$, for each root system Φ in the above table.

We already showed this in Section 6. Indeed, Lemmas 6.4 and 6.5 show that for all V_1 except those of A_1, A_2, D_4 or exceptional type, there exist u and v such that $\mathrm{tr}\,\big|_{V_1} o(v(u,v)) \neq 0$. But from equation 39, $\mathrm{tr}\,\big|_{V_1} o(v(u,v)) = \sum_{\alpha\in\Phi_{V_1}} P(\alpha)$, for some degree-four spherical harmonic P, and, because no Niemeier lattice root systems are of A_1, A_2, D_4 or exceptional type, we have the existence of the desired P for all the Niemeier lattice VOAs.

7.2 $v(u,v)$ is a highest-weight vector

To determine $v(u,v)$ in the general case we chose orthonormal vectors u and v in the Cartan subalgebra of V_1. In the lattice VOA case, the equivalent choice is that of α_1 and α_2 in the orthonormal basis for $\mathfrak{h} = L\otimes_{\mathbb{Z}}\mathbb{C}$. In fact, for the Niemeier lattice VOAs, $\alpha_1(-1)$ and $\alpha_2(-1)$ are orthonormal elements of $\mathrm{Span}\{\alpha_i(-1)\}$, a Cartan subalgebra of V_1. In the general case, we defined $v(u,v)$ to be $u[-1]^3u - 6u[-1]^2v[-1]v + v[-1]^3v$, which is in $V_{[4]}$. Whereas for V_L, the analogous vector is $v_P = \alpha_1(-1)^4 - 6\alpha_1(-1)^2\alpha_2(-1)^2 + \alpha(-1)^4$, which is a highest-weight vector in V_4. This brings up the question: Is $v(u,v)$ a highest-weight vector in the general case?

Proposition 7.2. *Let V be a VOA of strong CFT-type, let u and v be orthonormal elements of the Cartan subalgebra of V_1, and let $v(u,v) := u[-1]^3u - 6u[-1]^2v[-1]v + v[-1]^3v$. Then $v(u,v)$ is a highest-weight vector in V_4.*

Proof. We will show that $L[1]v(u,v) = 0$ and $L[2]v(u,v) = 0$. From Lemma 4.1, this suffices. Let a and b be in V_1. As V is of strong CFT-type b is a highest-weight vector. So $L[1]b = L[2]b = 0$. Using this and $[L[m], u[n]] = -nu[m+n]$, equation (7), we compute

$$\begin{aligned} L[1]a[-1]^2b[-1]b &= a[0]a[-1]b[-1]b + a[-1]a[0]b[-1]b + a[-1]^2b[0]b, \\ L[2]a[-1]^2b[-1]b &= a[1]a[-1]b[-1]b + a[-1]a[1]b[-1]b + a[-1]^2b[1]b. \end{aligned} \tag{42}$$

As we noted in the proof of Proposition 5.1, for VOAs of CFT-type and u, $v \in V_1$, $u[0]v = u(0)v = [u,v]$ and $u[1]v = u(1)v = \langle u|v\rangle$.

Assume that a and b commute. The affine Lie algebra commutation relations become

$$[a[m], b[n]] = \langle a|b\rangle m\delta_{m+n,0}\,\mathrm{id}\,.$$

In particular, $a[0]$ commutes with $a[-1]$ and $b[-1]$; so $a[0]a[-1]b[-1]b = a[-1]b[-1]a[0]b$. Furthermore, $a[0]b = [a, b] = 0$; so $a[0]a[-1]b[-1]b = 0$. Similarly, the other terms in the expansion of $L[1]a[-1]^2b[-1]b$ equal zero. So $L[1]a[-1]^2b[-1]b = 0$. Notice that $v(u, v)$ is a linear combination of $a[-1]^2b[-1]b$'s with $a = b = u$, $a = u$ and $b = v$, or $a = b = v$. Thus $L[1]v(u, v) = 0$.

For $L[2]a[-1]^2b[-1]b$,

$$\begin{aligned}
a[-1]a[1]b[-1]b &= a[-1]\langle a|b\rangle b + a[-1]b[-1]a[1]b,\\
&= \langle a|b\rangle a[-1]b + a[-1]b[-1]\langle a|b\rangle\mathbf{1},\\
&= 2\langle a|b\rangle a[-1]b,\\
a[1]a[-1]b[-1]b &= \langle a|a\rangle b[-1]b + a[-1]a[1]b[-1]b,\\
&= \langle a|a\rangle b[-1]b + 2\langle a|b\rangle a[-1]b,
\end{aligned}$$

and

$$a[-1]^2b[0]b = \langle b|b\rangle a[-1]a.$$

Thus from equation (42),

$$L[2]a[-1]^2b[-1]b = \langle a|a\rangle b[-1]b + 4\langle a|b\rangle a[-1]b + \langle b|b\rangle a[-1]a.$$

Applying this to our orthonormal u and v,

$$L[2]v(u, v) = 6u[-1]u - 6(u[-1]u + v[-1]v) + 6v[-1]v = 0.$$

□

8 Trace of $\nu(\alpha)$

In this section we define another vector in $V_{[4]}$ with zero trace on V_0 and show that its trace on V_1 is non-zero for those V excluded in the statement of Proposition 6.6. We use the following proposition of Dong and Mason [DMa].

Proposition 8.1. *Let V be a strongly holomorphic VOA of CFT-type with central charge c equal to* 8*,* 16 *or* 24*. Then for u, v in V_1,*

$$\kappa(u,v) = -\frac{1}{12}\langle u|v\rangle\big(b(c/2+2) + c + 24 - \dim V_1\big).$$

Here $b(k) = -\frac{2k}{B_k}$, where B_k is the kth Bernoulli number.

We specialize to $c = 24$, in which case $b(24/2+2) = b(14) = -24$, and we let n denote the dimension of V_1, for $n \neq 24$ this gives

$$\langle u|v\rangle = \frac{12}{n-24}\kappa(u,v).$$

We can use this equation to write the traces of $a[-1]b[-1]c[-1]d$ on V_0 and V_1 from Proposition 5.3 in terms of the Killing form. We get

$$\begin{aligned}\operatorname{tr}\Big|_{V_0} o(a[-1]b[-1]c[-1]d) &= -\frac{1}{60(n-24)}\big(4\kappa([a,b],[c,d]) + 5\kappa([a,d],[b,c])\big) \\ &\qquad + \frac{1}{8(n-24)^2}\sum_{\pi\in S_4}\kappa(\pi(a),\pi(b))\kappa(\pi(c),\pi(d)),\end{aligned} \tag{43}$$

$$\begin{aligned}\operatorname{tr}\Big|_{V_1} o(a[-1]b[-1]c[-1]d) &= -\frac{n+240}{60(n-24)}\big(4\kappa([a,b],[c,d]) + 5\kappa([a,d],[b,c])\big) \\ - \frac{n}{8(n-24)^2}\sum_{\pi\in S_4}&\kappa(\pi(a),\pi(b))\kappa(\pi(c),\pi(d)) + \frac{1}{6}\sum_{\pi\in S_3}\operatorname{tr}\Big|_{V_1} a(0)\pi\big(b(0)\big)\pi\big(c(0)\big)\pi\big(d(0)\big).\end{aligned} \tag{44}$$

If V_1 is abelian then the above traces are clearly zero for any choice of a, b, c and d. Hence we assume that V_1 is semi-simple. Let V_1 have Cartan decomposition $V_1 = \mathfrak{h}\oplus\bigoplus_{\alpha\in\Phi} L_\alpha$. For each root $\alpha\in\Phi$, and non-zero vector $x\in L_\alpha$ there exists $y\in L_{-\alpha}$ such that x, y and

$h = [x, y]$ are generators for a subalgebra of V_1 isomorphic to $\mathfrak{sl}_2$, with x, y and h satisfying the standard commutation relations, $[h, x] = 2x, [h, y] = -2y$, [Hum78, §8]. We call this the $\mathfrak{sl}_2$-subalgebra associated to α.

We want to compute the traces of $o(x[-1]y[-1]x[-1]y)$ on V_0 and V_1. Let $a = c = x$ and $b = d = y$. Plugging into equation (43) gives

$$\operatorname{tr}\Big|_{V_0} o(x[-1]y[-1]x[-1]y) = \frac{\kappa(h,h)}{60(n-24)} + \frac{1}{8(n-24)^2}\big(8\kappa(x,x)\kappa(y,y) + 16\kappa(x,y)^2\big).$$

Since $\kappa(\ ,\ )$ is invariant,

$$2\kappa(x,y) = \kappa([h,x],y) = \kappa(h,[x,y]) = \kappa(h,h).$$

Because $x(0)^2 : L_\beta \to L_{\beta+2\alpha}$, $\operatorname{tr}\big|_{V_1} x(0)^2 = \kappa(x,x) = 0$. Likewise, $\kappa(y,y) = 0$. Therefore

$$\operatorname{tr}\Big|_{V_0} o(x[-1]y[-1]x[-1]y) = \frac{\kappa(h,h)}{60(n-24)} + \frac{\kappa(h,h)^2}{2(n-24)^2}. \tag{45}$$

Plugging into equation (44) gives

$$\begin{aligned}\operatorname{tr}\Big|_{V_1} & o(x[-1]y[-1]x[-1]y) = \\ & \frac{\kappa(h,h)(n+240)}{60(n-24)} - \frac{n\kappa(h,h)^2}{2(n-24)^2} + \frac{1}{6}\operatorname{tr}\Big|_{V_1}\big(2x(0)y(0)x(0)y(0) + 4x(0)^2y(0)^2\big).\end{aligned} \tag{46}$$

To simplify this we need the following lemma.

Lemma 8.2. *Let α be a root of V_1 and let x, y and h be standard generators for the $\mathfrak{sl}_2$-subalgebra associated to α, $\mathfrak{a}$. Then*

$$\operatorname{tr}\Big|_{V_1}\big(2x(0)y(0)x(0)y(0) + 4x(0)^2y(0)^2\big) = \operatorname{tr}\Big|_{V_1} h(0)^4.$$

Proof. Consider V_1 as a module for $\mathfrak{a}$ under the adjoint action. Let $V_1 = M_1 \oplus M_2 \oplus \cdots \oplus M_m$ be a decomposition into irreducible $\mathfrak{a}$-modules. Because V_1 is a Lie algebra, $\dim M_i \leq 4$. The actions of x, y and h split over this decomposition so we are reduced to showing that

$$\mathrm{tr}\Big|_M \big(2x(0)y(0)x(0)y(0) + 4x(0)^2y(0)^2\big) = \mathrm{tr}\Big|_M h(0)^4, \tag{47}$$

for the irreducible $\mathfrak{sl}_2$-modules M of dimension one, two, three and four.

The irreducible modules of $\mathfrak{sl}_2$ are well known. The one-dimensional module is trivial, so all of the traces are zero and equation (47) is satisfied. In general, if M is $n+1$-dimensional, and v is a highest-weight vector in M. Then $\{v, y(0)v, y(0)^2v, \ldots, y(0)^n v\}$ is a basis for M, and each $y(0)^i v$ is an eigenvector for $h(0)$ with eigenvalue $n-2i$.

Let M be two-dimensional. The eigenvalues for the action of $h(0)$ are 1 and -1, so $\mathrm{tr}\big|_M h(0)^4 = 1^4 + (-1)^4 = 2$. For the $x(0)y(0)x(0)y(0)$ term,

$$\begin{aligned} x(0)y(0)v &= [x,y]v + y(0)x(0)v = h(0)v + y(0)0 = v, \\ x(0)y(0)y(0)v &= 0. \end{aligned} \tag{48}$$

So $\mathrm{tr}\big|_M x(0)y(0)x(0)y(0) = 1$. Equation (48) also implies that $\mathrm{tr}\big|_M x(0)^2y(0)^2 = 0$. Equation (47) is now easily verified.

For the M three-dimensional case, the eigenvalues of $h(0)$ are 2, 0 and -2. So $\mathrm{tr}\big|_M h(0)^4 = 2^4 + 0^4 + (-2)^4 = 32$.

$$\begin{aligned} x(0)y(0)v &= h(0)v = 2v \\ x(0)y(0)y(0)v &= h(0)y(0)v + y(0)x(0)y(0)v = 0 + y(0)2v = 2y(0)v \\ x(0)y(0)y(0)^2v &= 0 \end{aligned}$$

So $\mathrm{tr}\big|_M x(0)y(0)x(0)y(0) = 8$. For $x(0)^2y(0)^2$,

$$x(0)^2y(0)^2v = x(0)h(0)y(0)v + x(0)y(0)x(0)y(0)v = 0 + 2x(0)y(0)v = 4v,$$

and $x(0)^2y(0)^2$ acts as zero on the other basis elements. Thus $\operatorname{tr}\big|_M x(0)^2y(0)^2 = 4$. Equation (47) follows as $2 \cdot 8 + 4 \cdot 4 = 32$.

The four-dimensional case is verified the same way. We find that the trace of $h(0)^4$ is 164, the trace of $x(0)y(0)x(0)y(0)$ is 34 and the trace of $x(0)^2y(0)^2$ is 24. These values also satisfy equation (47) □

Using Lemma 8.2 to simplify equation (46) gives

$$\operatorname{tr}\Big|_{V_1} o(x[-1]y[-1]x[-1]y) = \frac{\kappa(h,h)(n+240)}{60(n-24)} - \frac{n\kappa(h,h)^2}{2(n-24)^2} + \frac{1}{6}\operatorname{tr}\Big|_{V_1} h(0)^4. \tag{49}$$

Next we compute the traces of $o(h[-1]^3h)$ on V_0 and V_1. From equations (43) and (44) we get

$$\operatorname{tr}\Big|_{V_0} o(h[-1]^3h) = \frac{3\kappa(h,h)^2}{(n-24)^2}, \tag{50}$$

$$\operatorname{tr}\Big|_{V_1} o(h[-1]^3h) = -\frac{3n\kappa(h,h)^2}{(n-24)^2} + \operatorname{tr}\Big|_{V_1} h(0)^4. \tag{51}$$

Define C to be $\frac{\operatorname{tr}|_{V_0} o(x[-1]y[-1]x[-1]y)}{\operatorname{tr}|_{V_0} o(h[-1]^3h)}$. From equations (45) and (50),

$$C = \frac{n-24}{180\kappa(h,h)} + \frac{1}{6}.$$

Define $\nu(\alpha)$ to be

$$\nu(\alpha) = x[-1]y[-1]x[-1]y - Ch[-1]^3h,$$

where x, y and h are standard generators for the $\mathfrak{sl}_2$-subalgebra associated to α.

Proposition 8.3. *Let V be a strongly holomorphic $c = 24$ VOA. Then*

$$\operatorname{tr}\Big|_{V_0} o(\nu(\alpha)) = 0,$$

$$\operatorname{tr}\Big|_{V_1} o(\nu(\alpha)) = \frac{\kappa(h,h)(n+120)}{30(n-24)} - \frac{n-24}{180\kappa(h,h)}\operatorname{tr}\Big|_{V_1} h(0)^4.$$

Proof. The first equation follows directly from the definitions of C and $\nu(\alpha)$. For the second, refer to equations (49) and (51) observe that

$$\begin{aligned}\frac{\kappa(h,h)(n+240)}{60(n-24)} - \frac{n\kappa(h,h)^2}{2(n-24)^2} + \left(\frac{n-24}{180\kappa(h,h)} + \frac{1}{6}\right)\frac{3n\kappa(h,h)^2}{(n-24)^2} \\ = \frac{\kappa(h,h)(n+240)}{60(n-24)} - \frac{n\kappa(h,h)^2}{2(n-24)^2} + \frac{n\kappa(h,h)}{60(n-24)} + \frac{n\kappa(h,h)^2}{2(n-24)^2},\end{aligned}$$

which equals $\frac{\kappa(h,h)(n+120)}{30(n-24)}$. Also,

$$\frac{1}{6}\operatorname{tr}\Big|_{V_1} h(0)^4 - \left(\frac{n-24}{180\kappa(h,h)} + \frac{1}{6}\right)\operatorname{tr}\Big|_{V_1} h(0)^4 = -\frac{n-24}{180\kappa(h,h)}\operatorname{tr}\Big|_{V_1} h(0)^4.$$

$\square$

Let $V_1 = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$ be a Lie-algebra decomposition of V_1 into simple Lie algebras. The $\mathfrak{sl}_2$-subalgebra generated by x, y and h is contained in one of the $\mathfrak{g}_i$'s. Because $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ for $j \neq i$, the values of $\kappa(h,h) = \operatorname{tr}\big|_{V_1} h(0)^2$ and $\operatorname{tr}\big|_{V_1} h(0)^4$ depend only on $\mathfrak{g}_i$. If x, y and h generate the $\mathfrak{sl}_2$-subalgebra associated to the root α in Φ, the root system of $\mathfrak{g}_i$, then

$$\begin{aligned}\kappa(h,h) &= \sum_{\beta\in\Phi} \beta(h)^2 = \frac{4}{(\alpha,\alpha)^2}\sum_{\beta\in\Phi}(\beta,\alpha)^2, \\ \operatorname{tr}\Big|_{V_1} h(0)^4 &= \sum_{\beta\in\Phi} \beta(h)^4 = \frac{16}{(\alpha,\alpha)^4}\sum_{\beta\in\Phi}(\beta,\alpha)^4.\end{aligned}$$

The above inner-product on $\mathfrak{h}^*$ is invariant under the action of the Weyl group and the Weyl group acts transitively on the roots of each length. Thus it suffices to compute $\kappa(h,h)$ and $\operatorname{tr}\Big|_{V_1} h(0)^4$ for one root of each length for each simple Lie algebra. We compute them using the root systems given in Appendix A and present them in Table 1.

In Section 6 we showed the existence of a vector with graded trace equal to $E_4(\tau)$ for all $c = 24$ strongly holomorphic VOAs with V_1 semisimple, *except those with* V_1 *a simple Lie algebra of type* A_1, A_2, D_4 *or exceptional.* The following proposition covers those exceptions.

Proposition 8.4. *Let* V *be a strongly holomorphic* $c = 24$ *VOA. If* V_1 *is a simple Lie algebra*

Table 1: $\kappa(h,h)$ and $\operatorname{tr}\big|_{V_1} h(0)^4$ for all roots

Φ	$\dim \mathfrak{g}_i$	α	$\kappa(h,h)$	$\operatorname{tr}\big|_{V_1} h(0)^4$
A_ℓ	$\ell^2+2\ell$	—	$4\ell+4$	$4\ell+28$
B_ℓ	$2\ell^2+\ell$	long	$8\ell-4$	$8\ell+20$
		short	$16\ell-8$	$64\ell-32$
C_ℓ	$2\ell^2+\ell$	long	$4\ell+4$	$4\ell+28$
		short	$8\ell+8$	$8\ell+80$
D_ℓ	$2\ell^2-\ell$	—	$8\ell-8$	$8\ell+16$
E_6	78	—	48	72
E_7	133	—	72	96
E_8	248	—	120	144
F_4	52	long	36	60
		short	72	240
G_2	14	long	16	40
		short	48	360

of type A_1, A_2, D_4, E, F or G, then there exists a vector $v \in V_{[4]}$ with graded trace equal to $E_4(\tau)$.

Proof. Choose α a long root of V_1 and consider $\nu(\alpha)$. Since V is of strong CFT-type, $\nu(\alpha) \in V_{[4]}$. So $Z(\nu(\alpha),\tau)$ has weight four. By Proposition 8.3, $o(\nu(\alpha))$ has zero trace on V_0, so $Z(\nu(\alpha),\tau)$ is holomorphic. The trace of $o(\nu(\alpha))$ on V_1 depends on $\kappa(h,h)$, $\operatorname{tr}\big|_{V_1} h(0)^4$ and n, the dimension of V_1. Because we assume that V_1 is simple, $V_1 = \mathfrak{g}_i$ and $n = \dim \mathfrak{g}_i$. To complete the proof, we insert the values from Table 1 into the result of Proposition 8.3 to compute the traces of $o(\nu(\alpha))$ on V_1, for V_1 of types $A_1, A_2, D_4, E_6, E_7, E_8, F_4$ and G_2 and α a long root. The results of these computations are given in Table 2 Because $\operatorname{tr}\big|_{V_1} o(\nu(\alpha)) \neq 0$, $Z(\nu(\alpha),\tau)$ is non-zero. Since the space of holomorphic weight-four modular forms is $\mathbb{C}E_4(\tau)$, some multiple of $\nu(\alpha)$ has graded trace $E_4(\tau)$. □

Propositions 6.6 and 8.4 imply that all strongly holomorphic, $c = 24$ VOAs with V_1 semisimple contain an element of square-bracket weight four, with graded trace $E_4(\tau)$. By Proposition 3.3, this implies the existence of a highest-weight vector with graded trace $E_4(\tau)$. Because the δ-ideal generated by $E_4(\tau)$ is the space of all modular forms, the main theorem, Theorem 3.1, now follows from Proposition 3.2, of Dong and Mason.

Table 2: Trace of $\nu(\alpha)$ on V_1

V_1	C	$\mathrm{tr}\big\|_{V_1} o(\nu(\alpha))$
$A_1(\mathbb{C})$	$\frac{73}{480}$	$-\frac{71}{35}$
$A_2(\mathbb{C})$	$\frac{43}{270}$	$-\frac{44}{15}$
$D_4(\mathbb{C})$	$\frac{181}{1080}$	$\frac{266}{9}$
$E_6(\mathbb{C})$	$\frac{83}{480}$	$\frac{65}{12}$
$E_7(\mathbb{C})$	$\frac{2269}{12{,}960}$	$\frac{70{,}091}{14{,}715}$
$E_8(\mathbb{C})$	$\frac{239}{1350}$	$\frac{2666}{525}$
$F_4(\mathbb{C})$	$\frac{277}{1620}$	$\frac{6721}{945}$
$G_2(\mathbb{C})$	$\frac{47}{288}$	$-\frac{6307}{900}$

Remark 8.5. *For any V_1 and any long root α, Table 1 shows that $\mathrm{tr}\big|_{V_1} h(0)^4 = \kappa(h,h) + 24$. It turns out that $\mathrm{tr}\Big|_{V_1} o(\nu(\alpha)) = 0$ if and only if $\dim V_1 = 6(\kappa(h,h) + 24)$.*

9 Conclusion

In conclusion, we discuss the implications of our result in other cases. Notice that Propositions 5.3 and 6.1 apply to all strongly rational VOAs. In fact, in all of Section 6, we only use the assumption that V is holomorphic to guarantee that the graded trace of $v(u,v)$ is a multiple of $E_4(\tau)$. For strongly rational VOAs, we know that V_1 is reductive [DMb]. Let V_1 have decomposition

$$V_1 = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m \oplus \mathfrak{h}, \tag{52}$$

such that each $\mathfrak{g}_i$ is simple and $\mathfrak{h}$ is abelian. Proposition 6.1 and Lemmas 6.4 and 6.5 imply that: If $m > 1$, or $m = 1$ and $\mathfrak{g}_1$ is not of type A_1, A_2, D_4 or exceptional, then there exist u and v in V_1 such that

$$Z(v(u,v),\tau) = q^{-c/24} \sum_{n \geq 1} \mathrm{tr}\Big|_{V_n} o(v(u,v)) q^n,$$

and the coefficient of $q^{1-c/24}$ is non zero.

Assume that V is holomorphic. This implies that the graded traces are meromorphic modular forms. For holomorphic VOAs, the central charge, c, is positive and divisible by eight. If $c \leq 24$, then $\operatorname{tr}\big|_{V_0} o(v) = 0$ implies that $Z(v, \tau)$ is holomorphic. We explore the implications of our work in the $c = 8$ and $c = 16$ cases. Unlike the $c = 24$ case, for these central charges the graded traces have non-trivial characters.

Let $\eta(\tau) := q^{1/24} \prod_{n>0}(1 - q^n)$. It is modular of weight one-half with a non-trivial character [Lan76]. For $c = 8$, we can eliminate the character of $Z(v, \tau)$ by multiplying it by $\eta(\tau)^8$; $Z(v, \tau)\eta(\tau)^8$ is a modular form of weight $4 + [\mathrm{wt}]v$, and we compute

$$\begin{aligned} \eta(\tau)^8 Z(v, \tau) &= q^{1/3} \prod (1 - q^n)^8 q^{-1/3} \sum_{n \geq 0} \operatorname{tr}\Big|_{V_n} o(v) q^n, \\ &= \operatorname{tr}\Big|_{V_0} o(v) + \big(\operatorname{tr}\Big|_{V_1} o(v) - 8 \operatorname{tr}\Big|_{V_0} o(v)\big) q + \dots. \end{aligned}$$

Thus $\eta(\tau)^8 Z(v, \tau)$ is holomorphic, has no character and is a cusp form, if, and only if $\operatorname{tr}\big|_{V_0} o(v) = 0$. Hence for $c = 8$, the results of Section 6 define an element $v(u, v) \in V_{[4]}$ so that $\eta(\tau)^8 Z(v(u, v), \tau)$ is a cusp form of weight eight. There are no non-zero cusp forms of weight eight, so this implies that $Z(v(u, v), \tau) = 0$ for all choices of u and v. The contrapositives of Lemmas 6.4 and 6.5 then imply that $V_1 = \mathfrak{g}_1 \oplus \mathfrak{h}$ and either $\mathfrak{g}_1$ is of type A_1, A_2, D_4 or exceptional, or $\mathfrak{g}_1 = 0$. In fact more is known, Dong and Mason have shown that the only strongly holomorphic $c = 8$ VOA is the lattice VOA of the E_8 root lattice [DMa]. For this VOA, V_1 is a Lie algebra of type E_8. This agrees with our result.

For $c = 16$, we can show similarly that $\eta(\tau)^{16} Z(v, \tau)$ is a holomorphic modular form without character of weight $8 + [\mathrm{wt}]v$, which is a cusp form, if, and only if $\operatorname{tr}\big|_{V_0} o(v) = 0$. So $\eta(\tau)^{16} Z(v(u, v), \tau)$ is a cusp form of weight 12. The space of cusp forms of weight 12 is $\mathbb{C}\Delta$, where $\Delta = \eta(\tau)^{24}$. Dong and Mason have shown that the only strongly holomorphic $c = 16$ VOAs are the lattice VOAs of the root lattices $E_8 + E_8$ and D_{16} [DMa]. For the lattice VOA $V_{E_8+E_8}$, V_1 is of type $E_8 + E_8$, and Proposition 6.4 implies that there exist u and v in V_1 such that

$$Z(v(u,v),\tau) = \frac{c\Delta}{\eta(\tau)^{16}},$$

for some $c \neq 0$. For the lattice VOA $V_{D_{16}}$, V_1 is of type D_{16}, and Proposition 6.5 gives the same result.

The calculations of Section 8 use the assumption that V is strongly holomorphic and $c = 24$ in order to eliminate the bilinear form $\langle \mid \rangle$ form the formulas for the traces of $o(a[-1]b[-1]c[-1]d)$, equations 43 and 44. However, for strongly holomorphic V with $c = 8$ or $c = 16$, Dong and Mason's proposition, Proposition 8.1, also yields an equation relating the Killing form on V_1 to the from $\langle \mid \rangle$. Thus in those cases we can make calculations analogous to those of Section 8 to define a vector v like $\nu(\alpha)$ in $V_{[4]}$ with $\operatorname{tr}\big|_{V_0} o(v) = 0$ and compute the trace of $o(v)$ on V_1. For $c = 16$, if $\operatorname{tr}\big|_{V_1} o(v) \neq 0$, this would simply replicate the result we just stated for $v(u,v)$, and if $\operatorname{tr}\big|_{V_1} o(v) = 0$, then $Z(v,\tau) = 0$. For $c = 8$, however, we have shown that $Z(v,\tau) = 0$ for all $v \in V_{[4]}$. So pursuing calculations like those of Section 8 should always produce $\operatorname{tr}\big|_{V_1} o(v) = 0$. This provides a valuable check of the computations in Sections 5 and 8.

Let V be a strongly holomorphic $c = 8$ VOA. Then $b(8/2+2) = b(6) = -504$, and, because $V = V_{E_8}$, the dimension of V_1 is 248. So from Proposition 8.1

$$\kappa(u,v) = 60\langle u|v\rangle.$$

Let x, y and h be standard generators for the $\mathfrak{sl}_2$-subalgebra of V_1 associated to a root α of V_1. Because V_1 is of type E_8 there is only one root length and all choices of α give the same results. Making the same calculations as in the $c = 24$ case:

$$\operatorname{tr}\Big|_{V_0} o(h[-1]h) = \frac{1}{12}, \qquad \operatorname{tr}\Big|_{V_1} o(h[-1]h) = \frac{122}{3},$$
$$\operatorname{tr}\Big|_{V_0} o(x[-1]y[-1]x[-1]y) = \frac{1}{60}, \qquad \operatorname{tr}\Big|_{V_1} o(x[-1]y[-1]x[-1]y) = \frac{122}{15}.$$

Define $\nu(\alpha) := 5x[-1]y[-1]x[-1]y - h[-1]^3h$. Then the traces of $o(\nu(\alpha))$ on V_0 and V_1 are both zero, which is what we expected.

In order to carry out the calculations of Section 8, we need V_1 to have a semisimple part, so that a root α exists, i.e., we need $m \geq 1$ in the decomposition of V_1, equation (52). Additionally, for $c = 24$, we need $\dim V_1 \neq 24$. However this is implied by the assumption that $m \geq 1$. Indeed, for $c = 24$, Proposition 8.1 gives $(\dim V_1 - 24)\langle\ |\ \rangle = 12\kappa(\ ,\ )$, and the Killing form, $\kappa(\ ,\ )$, is non-degenerate on semisimple Lie algebras. Thus Remark 8 says that: For a strongly holomorphic $c = 24$ VOA, if $m \geq 1$, there exists a root α such that $Z(\nu(\alpha), \tau) = cE_4(\tau)$, unless $\dim V_1 = 6(\kappa(h,h) + 24)$.

Dong and Mason have proven that if V is a strongly holomorphic $c = 24$ VOA, then V_1 is either semisimple, abelian or zero [DMa]. Our main result applies to the V_1 semisimple case. For $V_1 = 0$, it is conjectured that V is the moonshine module. In this case the constant term of the graded traces are always zero, so there is no element of the moonshine module with graded trace equal to $E_4(\tau)$. Dong and Mason have described the space of graded traces of the moonshine module [DM00]. This leaves the case of abelian V_1.

The techniques we used for V_1 semisimple will not work for V_1 abelian. Indeed, in the abelian case our basic calculation of the traces of $o(a[-1]b[-1]c[-1]d)$, Proposition 5.3, gives

$$\begin{aligned}\operatorname{tr}\Big|_{V_0} o(a[-1]b[-1]c[-1]d) &= \frac{1}{1152}\sum_{\pi \in S_4} \langle \pi(a)|\pi(b)\rangle\langle \pi(c)|\pi(d)\rangle,\\ \operatorname{tr}\Big|_{V_1} o(a[-1]b[-1]c[-1]d) &= \frac{n-48}{1152}\sum_{\pi \in S_4} \langle \pi(a)|\pi(b)\rangle\langle \pi(c)|\pi(d)\rangle.\end{aligned}$$

So any linear combination v of these elements such that the trace of $o(v)$ on V_0 is zero, also has the trace of $o(v)$ on V_1 zero.

Because it is true for all other strongly holomorphic $c = 24$ VOAs with $V_1 \neq 0$, we suspect that for V_1 abelian there exists a element v with $Z(v, \tau) = E_4(\tau)$. There are a number of possible approaches to proving this. Proposition 5.1 does not apply to V_1 abelian, so one could consider other square-bracket-weight-four elements of $A(V_{[1]})(V_1 \oplus \mathbf{1})$. If this fails, one could try using the vertex operators of elements of V_2. Dong and Mason have shown that the only strongly holomorphic $c = 24$ VOA with V_1 abelian is the Leech lattice VOA, V_Λ [DMa].

Therefore, one could explicitly use the structure of the Leech lattice, Λ. In particular the square-length four elements of Λ, which yield elements of $V_{\Lambda 2}$.

Part II

Moonshine module

10 Introduction

This paper explores certain functions associated to the moonshine module, $V^\natural$, a vertex-operator algebra (VOA). The existence of the moonshine module and some of its many remarkable properties were conjectured by Conway and Norton in their paper *Monstrous moonshine* [CN79]; "monstrous," because it is a $\mathbb{Z}$-graded module for the Fisher-Griess monster, the largest sporadic simple group. Write $V^\natural = \oplus_{n\geq 0} V_n$, and define the "graded dimension" of $V^\natural$ to be $q^{-1}\sum_{n\geq 0}\dim V_n q^n$. Frenkel, Lepowsky and Meurman constructed $V^\natural$ so that its graded dimension is the modular function $J(q) = q^{-1} + 196,884q + \cdots$, as conjectured by Conway and Norton [FLM88]. Monstrous moonshine also asserts that by generalizing the graded dimension with the introduction of a character of the monster, one obtains the q-expansions of other hauptmoduls. This was proven by Borcherds in 1992 [Bor92].

This paper concerns another generalization of the graded dimension. For a state $v \in V^\natural$, we define a $V^\natural$-endomorphism $o(v)$, which preserves the grading. The graded trace of v is defined to be

$$Z(v,q) := q^{-1}\sum_{n\geq 0} \operatorname{tr}\Big|_{V_n} o(v) q^n.$$

The graded dimension is the graded trace of the vacuum vector, for which $o(v) = \mathrm{id}$. Remarkably, the graded traces of certain elements of V_k, including all the so-called highest-weight vectors, are modular forms of weight k [Zhu96]. Graded traces are also of interest to Physicists studying Conformal Field Theory, who refer to them as one-point correlation functions [BPZ84]. That term is also more common than graded trace in the mathematics literature.

The immediate motivation for this paper is Dong and Mason's paper *Monstrous moon-*

shine of higher weight [DM00]. One of their results is the explicit calculation of the graded trace of the highest-weight vector $1 \otimes (e^{\lambda} + e^{-\lambda})$, where λ is some element of the Leech lattice, Λ [DM00, Theorem 3]. Each $1 \otimes (e^{\lambda} + e^{-\lambda})$ is a highest-weight vector in $V_{(\lambda,\lambda)/2}$, and, for $\lambda \neq 0$, its graded trace is a cusp form, which is non-zero if and only if $\lambda \in 2\Lambda$ and $(\lambda, \lambda) \geq 24$. In this paper, we define a another family of highest-weight vectors, $v(\lambda)$ (see equation (59)) and compute their graded traces (see Theorem 13.1). Each $v(\lambda)$ is in $V_{(\lambda,\lambda)/2+2}$ and its graded trace is also a cusp form, which is non-zero if and only if $0 \neq \lambda \in 2\Lambda$ and $(\lambda, \lambda) \geq 32$.

The Leech lattice is even, so for $\lambda \in 2\Lambda$, $8|(\lambda, \lambda)$. Thus Dong and Mason's paper shows that for all $k \equiv 0 \mod 4$ and at least 12, there exists a highest-weight vector v in V_k so that $Z(v, q)$ is a non-zero cusp form (of weight k). This paper gives the same result for $k \equiv 2 \mod 4$ and at least 18. Let M_k^0 be the space of weight-k cusp forms. We know that for $k \geq 0$,

$$\dim M_k^0 = \begin{cases} 0 & k \equiv 1 \mod 2, \\ \lfloor \frac{k}{12} \rfloor - 1 & k \equiv 2 \mod 12, \\ \lfloor \frac{k}{12} \rfloor & \text{otherwise}; \end{cases}$$

see Serre, for example [Ser73]. Thus for each $M_k^0 \neq 0$, we have a highest-weight vector with graded trace equal to some non-zero element of M_k^0. By averaging over the moonshine module, we get a *monster-invariant* highest-weight vectors with the same graded trace.

Harada and Lang found a generating function for the dimension of the space of monster-invariant highest-weight vectors in V_k, for each k [HL98]. Mason noticed that for the first few k these dimensions coincide with $\dim M_k^0$ and that they are always at least $\dim M_k^0$. He conjectures that for each cusp form f there is a monster-invariant vector v, such that $Z(v, q) = f$ [DM00, Conjecture (A)]. The combined results of this paper and Dong and Mason's paper prove this in the cases for which $\dim M_k^0 = 1$, that is for $k = 12, 16, 18, 20, 22$ or 26.

In a manner analogous to the way the Leech lattice is used to construct $V^{\natural}$, Dolan, Goddard and Montague use other even self-dual lattices to construct VOAs called $\mathbb{Z}_2$-orbifolds [DGM90]. Even self-dual lattices with rank 24, called Niemeier lattices, produce $\mathbb{Z}_2$-orbifolds with central charge 24. The calculations in this paper use the central charge but do not depend on any of the special properties of the Leech lattice. Therefore the main theorem also applies to the $\mathbb{Z}_2$-orbifolds of the other Niemeier lattices.

11 Notation and Definitions

We use the definition of vertex operator algebra given in *On Axiomatic Approaches to Vertex Operator Algebras and Modules* and also adopt most of their notation [FHL93]. A VOA is specified $(V, Y, \mathbf{1}, \omega)$; V is the underlying vector space; $Y(v,z) = \sum_{n\in\mathbb{Z}} v(n)z^{-n-1}$ is the vertex operator associated with v; $\mathbf{1}$ is the vacuum vector, and ω is the conformal vector. We denote the central charge c and let $V = \oplus_{n\geq n_0} V_n$ be the decomposition of V into homogeneous spaces. Vertex operator algebras are, by definition, representations of the Virasoro algebra via $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$. By a *highest-weight vector* v of V we mean a highest-weight vector of this representation, i.e., $L(n)v = 0$ for all $n > 0$.

One-point correlation functions are defined by Zhu [Zhu96]. We call them graded traces, because the term is shorter and more algebraic. Let $v \in V_k$ and define $o(v) = v(k-1)$. The linear map $o(v)$ preserves the grading of V and we define the *graded trace* to be

$$Z_V(v,q) = \mathrm{tr}_V\, o(v)q^{L(0)-c/24} := q^{-c/24}\sum_{n\in\mathbb{Z}} \mathrm{tr}\big|_{V_n} o(v)q^n. \tag{53}$$

We also use the notation $Z_V(v,\tau)$; τ is related to q via $q = e^{2\pi i\tau}$.

The moonshine module, $V^\natural$, is a VOA constructed using the Leech lattice VOA, V_Λ. In our calculations of graded traces we will need explicit formulas for the vertex operators of V_Λ. To this end we summarize the definition of lattice VOA below. (For more details, see [Don93].) Further aspects of the definition of the moonshine module will be presented in the text as needed. (For the complete construction of $V^\natural$, see [FLM88].)

Let L be a positive-definite even integral lattice with inner product $(\ ,\ )$ and let $\mathfrak{h} :=$

$L \otimes_{\mathbb{Z}} \mathbb{C}$. The corresponding Heisenberg Lie algebra is $\hat{\mathfrak{h}}_{\mathbb{Z}} := \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$, and $M(1) := U(\hat{\mathfrak{h}}_{\mathbb{Z}}) \otimes_{U(\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}c)} M$ is the irreducible $\hat{\mathfrak{h}}_{\mathbb{Z}}$-module induced from the one-dimensional $\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}c$-module M, on which elements of $\mathfrak{h} \otimes \mathbb{C}[t]$ act as 0, and c acts as the identity. Let $\{\alpha_1, \alpha_2, ..., \alpha_m\}$ be a basis for $\mathfrak{h}$, and denote the element $\alpha \otimes t^n$ of $\hat{\mathfrak{h}}_{\mathbb{Z}}$ by $\alpha(n)$. Polynomials in the variables $\alpha_i(-n)$, where $1 \leq i \leq m$ and $n \geq 1$, form a basis of $M(1)$.

Since L is a free abelian group of finite rank, we can define a $\mathbb{Z}$-bilinear cocycle $\epsilon : L \times L \to \mathbb{Z}/2\mathbb{Z}$ such that $\epsilon(\alpha, \beta) - \epsilon(\beta, \alpha) \equiv (\alpha, \beta) \mod 2$. Let $\mathbb{C}\{L\}$ be the twisted group algebra with basis $\{e^\beta | \beta \in L\}$ and multiplication $e^\alpha e^\beta = (-1)^{\epsilon(\alpha,\beta)} e^{\alpha+\beta}$. The lattice VOA, V_L, is defined to be $M(1) \otimes \mathbb{C}\{L\}$. It is linearly isomorphic to $S(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes \mathbb{C}[L]$.

Let $v = \alpha_1(-n_1)\alpha_2(-n_2) \dots \alpha_k(-n_k) \otimes e^\beta$ be an element of V_L.

$$\begin{multline} Y(v, z) = {}^{\circ}_{\circ}\left(\frac{1}{(n_1 - 1)!} \left(\frac{\partial}{\partial z} \right)^{n_1 - 1} \alpha_1(z) \right) \dots \\ \left(\frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial z} \right)^{n_k - 1} \alpha_k(z) \right) E^-(-\beta, z) E^+(-\beta, z) e^\beta z^\beta {}^{\circ}_{\circ} \quad (54) \end{multline}$$

where

$$\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha(n) z^{-n-1},$$
$$E^{\pm}(\beta, z) = \exp\left(\sum_{n \in \mathbb{N}} \frac{\beta(\pm n)}{\pm n} z^{\mp n} \right).$$

The actions of $\alpha(n)$ on $u \otimes e^\gamma \in V_L$ are

$$\alpha(n) \cdot u \otimes e^\gamma = \begin{cases} (\alpha(n)u) \otimes e^\gamma & \text{if } n < 0, \\ n \frac{\partial u}{\partial(\alpha(-n))} \otimes e^\gamma & \text{if } n > 0, \\ (\alpha, \gamma) u \otimes e^\gamma & \text{if } n = 0. \end{cases} \quad (55)$$

The actions of e^β and z^β are

$$e^\beta \cdot u \otimes e^\gamma = (-1)^{\epsilon(\beta,\gamma)} u \otimes e^{\beta+\gamma}, \quad (56)$$

$$z^\beta \cdot u \otimes e^\gamma = u \otimes e^\gamma z^{(\beta,\gamma)}.$$

The normal ordering $\overset{\circ}{\circ}\cdots\overset{\circ}{\circ}$ of the operators in $Y(v,z)$ indicates that $\alpha(n)$ with positive n acts before, i.e., to the right of, $\alpha(n)$ with negative n and that $\alpha(0)$ and z^β act before e^β.

In order to state our main theorem, we need to define some functions. The Dedekind eta function,

$$\eta(\tau) := q^{1/24} \prod_{n>0} (1-q^n),$$

is a weight one-half modular form, with a non-trivial character. The Jacobi theta functions are:

$$\begin{aligned}
\Theta_1(\tau) &:= 2q^{1/8} \prod_{n>0} (1-q^n)(1+q^n)^2 = 2\frac{\eta(2\tau)^2}{\eta(\tau)}, \\
\Theta_2(\tau) &:= \prod_{n>0} (1-q^n)(1-q^{n-1/2})^2 = \frac{\eta(\frac{\tau}{2})^2}{\eta(\tau)}, \\
\Theta_3(\tau) &:= \prod_{n>0} (1-q^n)(1+q^{n-1/2})^2 = \frac{\eta(\tau)^5}{\eta(2\tau)^2\eta(\frac{\tau}{2})^2}.
\end{aligned}$$

They are the Jacobi theta functions defined as in Chandrasekharan, with the first variable set equal to 0 [Cha85]. The “Eisenstein series” $E_2(\tau)$ is

$$E_2(\tau) := 1 - 24 \sum_{n>0} \sigma_1(n) q^n.$$

While it is defined by a q-expansion analogous to that of the Eisenstein series $E_{2k}(\tau)$, $k \geq 2$, it is not a modular form.

12 Highest-Weight Vectors

The moonshine module is the direct sum of two spaces. The first is a sub-VOA of the Leech Lattice VOA, V_Λ. It is denoted V_Λ^+. The second is a twisted module for V_Λ. It is denoted $(V_\Lambda^T)^+$. Because we will be directly computing graded traces, we limit our search for highest-weight vectors of $V^\natural$ to elements of V_Λ^+. The vertex operators associated with these elements are known explicitly, unlike those of $(V_\Lambda^T)^+$.

The conformal vector ω of $V^\natural$ is the same as that of V_Λ, namely $\frac{1}{2}\sum_{i=1}^{24}\alpha_i(-1)^2$, where $\{\alpha_i \mid 1 \leq i \leq 24\}$ is an orthonormal basis for $\mathfrak{h} = \Lambda \otimes_\mathbb{Z} \mathbb{C}$. The set of highest-weight vectors of $V^\natural$ in V_Λ^+ contains exactly those highest-weight vectors of V_Λ that lie in V_Λ^+. However, the graded traces of these vectors on the spaces $V^\natural$ and V_Λ differ, and there are highest-weight vectors with non-zero trace on $V^\natural$ and zero trace on V_Λ. In fact, it is not hard to show that any highest-weight vector of V_Λ that is not in $M(1)$ has a graded trace on V_Λ equal to zero. This is not necessarily true of its graded trace on $V^\natural$. Dong and Mason computed the graded trace of $1 \otimes (e^\lambda + e^{-\lambda})$ on $V^\natural$ for all $0 \neq \lambda \in 2\Lambda$ and found the graded trace to be non-zero for $(\lambda, \lambda) \geq 24$ [DM00].

First we examine V_Λ^+ more closely. It is the sub-VOA of $V_\Lambda = M(1) \otimes \mathbb{C}\{\Lambda\}$ fixed by the VOA automorphism t, which is defined by $t(e^\lambda) = e^{-\lambda}$ for $e^\lambda \in \mathbb{C}\{\Lambda\}$ and

$$t\big(\alpha_1(-n_1)\alpha_2(-n_2)...\alpha_k(-n_k)\big) = (-1)^k\big(\alpha_1(-n_1)\alpha_2(-n_2)...\alpha_k(-n_k)\big), \tag{57}$$

for $\alpha_1(-n_1)\alpha_2(-n_2)...\alpha_k(-n_k) \in M(1)$. For β in Λ, define

$$V(\beta) := M(1) \otimes (\mathbb{C}e^\beta + \mathbb{C}e^{-\beta}),$$

and $V(\beta)^+$ to be those elements of $V(\beta)$ fixed by t. Clearly, we have the following vector-space direct-sum decomposition;

$$V_\Lambda^+ = \bigoplus_{\beta \in \Lambda/<\pm 1>} V(\beta)^+,$$

in which $\bigoplus_{\beta \in \Lambda/<\pm 1>}$ means that the sum is taken over exactly one element from all pairs $\pm\beta$.

Lemma 12.1. *Let $v \in V(\beta)^+$. There exists a unique $u \in M(1)$ such that*

$$v = u \otimes e^\beta + t(u) \otimes e^{-\beta}.$$

Proof. Let $v \in V(\beta)^+$ and let $\{u_\alpha | \alpha \in A\}$ be a basis for $M(1)$. As $v \in V(\beta)$, we can write

$$v = \sum_{\alpha \in A} c_\alpha \big(u_\alpha \otimes e^\beta\big) + \sum_{\alpha \in A} d_\alpha \big(u_\alpha \otimes e^{-\beta}\big),$$

for some unique $c_\alpha, d_\alpha \in \mathbb{C}$. Define $u := \sum_{\alpha \in A} c_\alpha u_\alpha$ and $u' := \sum_{\alpha \in A} d_\alpha u_\alpha$. As $v \in V_\Lambda^+$, $v = t(v)$, i.e.,

$$u \otimes e^\beta + u' \otimes e^{-\beta} = t(u') \otimes e^\beta + t(u) \otimes e^{-\beta}.$$

The uniqueness of u' implies that $t(u) = u'$ and the result follows. □

We will also find it useful to know how the vertex operators act on the subspaces $V(\beta)^+$.

Lemma 12.2. *Let $\beta, \gamma \in \Lambda$ and $v \in V(\gamma)^+$. Then*

$$Y(v, z) : V(\beta)^+ \to \big(V(\beta + \gamma)^+ \cup V(\beta - \gamma)^+\big) [\![z, z^{-1}]\!].$$

Proof. Let $v \in V(\gamma)^+$ and $w \in V(\beta)^+$. For all $n \in \mathbb{Z}$,

$$v(n)w \in V(\beta + \gamma) \cup V(\beta - \gamma).$$

To see this, examine the vertex operator for V_Λ^+. It is the restriction of the vertex operator for V_Λ given in equation (54). (In particular, see equation (56).) Furthermore, since t is a VOA automorphism, $v(n)w \in V_\Lambda^+$. Hence

$$v(n)w \in (V(\beta + \gamma) \cup V(\beta - \gamma)) \cap V_\Lambda^+ = V(\beta + \gamma)^+ \cup V(\beta - \gamma)^+.$$

□

Let $v \in V_\Lambda^+$ be a highest-weight vector with decomposition $v = \sum_{\beta \in \Lambda / <\pm 1>} v_\beta$ such that

$v_\beta \in V(\beta)^+$. Since v is a highest-weight vector, for all $n > 0$

$$0 = L(n)v = \sum_{\beta \in \Lambda/<\pm 1>} L(n)v_\beta.$$

The conformal vector of $V^\natural$ is in $V(0)^+$. So by Lemma 12.2, $L(n)v_\beta \in V(\beta)^+$. The uniqueness of the decomposition, $0 = \sum_{\beta \in \Lambda/<\pm 1>} 0$, implies that $L(n)v_\beta = 0$ for all $n > 0$. Therefore v_β is a highest-weight vector. Thus finding the highest-weight vectors in every $V(\beta)^+$ would suffice to describe the highest-weight vectors in V_Λ^+.

Define $V(\beta)^+_n := \{u \otimes e^\beta + t(u) \otimes e^{-\beta} | \operatorname{wt} u = n\}$. Clearly $V(\beta)^+ = \bigoplus_{n \geq 0} V(\beta)^+_n$, and as $\operatorname{wt}(L(m)v) = \operatorname{wt} v - m$;

$$L(m) : V(\beta)^+_n \to \begin{cases} V(\beta)^+_{n-m} & \text{if } m \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

So if $v \in V(\beta)^+$ is a highest-weight vector, and $v = \sum_{n \geq 0} v_n$ with $v_n \in V(\beta)^+_n$, then v_n is a highest-weight vector for all n. Therefore, it would suffice to find the highest-weight vectors in all the $V(\beta)^+_n$s.

Our biggest challenge is not finding highest-weight vectors but finding highest-weight vectors with non-zero graded trace on $V^\natural$. The graded trace on $V^\natural$ is the sum of the graded traces on V_Λ^+ and $(V_\Lambda^T)^+$. To increase the odds of getting a non-zero trace on $V^\natural$, we assume that the graded trace on V_Λ^+ is non-zero.

Proposition 12.3. *If $v \in V(\lambda)^+$ and $Z_{V_\Lambda^+}(v, q) \neq 0$ then $\lambda \in 2\Lambda$. Furthermore, if $\lambda \neq 0$, then*

$$Z_{V_\Lambda^+}(v, q) = q^{(\lambda,\lambda)/8-1} \sum_{n \geq 0} \operatorname{tr}|_{V(\lambda/2)^+_n} o(v) q^n.$$

Proof. The VOA V_Λ^+ has central charge 24, and the weight of the elements of $V(\beta)^+_n$ is $(\beta, \beta)/2+$

n. Thus the graded trace, equation (53), on V_Λ^+ is

$$Z_{V_\Lambda^+}(v,q) = q^{-1} \sum_{\beta \in \Lambda/<\pm 1>} q^{(\beta,\beta)/2} \sum_{n \geq 0} \operatorname{tr}|_{V(\beta)_n^+} o(v) q^n. \tag{58}$$

By supposition, $Z_{V_\Lambda^+}(v,q) \neq 0$. Hence $\operatorname{tr}|_{V(\beta)_n^+} o(v) \neq 0$ for some $\beta \in \Lambda$. Therefore the intersection of $o(v)\,(V(\beta)^+)$ and $V(\beta)^+$ must be non-trivial. By Lemma 12.2, this implies that $\lambda = 0$ or $\lambda = \pm 2\beta$. In both cases $\lambda \in 2\Lambda$. Now assume that $\lambda \neq 0$. The only $V(\beta)^+$ which contributes to $Z_{V_\Lambda^+}(v,q)$ is $V(\lambda/2)^+$. The second assertion follows from equation (58). □

In this paper we treat the $\lambda \neq 0$ case; the calculations for $\lambda = 0$ are quite different. If $\lambda = 0$, the following computation of highest-weight vectors is changed as the length of λ, ℓ, equals zero. Additionally, for $\lambda = 0$, the the vertex operators of elements in $V(\lambda)^+$ include

$$E^-(-\lambda,\tau)E^+(-\lambda,\tau)e^\lambda z^\lambda = \mathrm{id},$$

significantly altering the graded trace.

For each $n \geq 0$, we would like to find the graded trace on $V^\natural$ of highest-weight vectors in $S_n := \{v \in V(\lambda)_n^+ | \lambda \neq 0, \lambda \in 2\Lambda\}$. The weight of an element of S_n is $(\lambda,\lambda)/2 + n$. So if n is odd, the weight of all $v \in S_n$ is also odd. The graded trace of a highest-weight vector v on $V^\natural$ is a modular form of level one and weight equal to the weight of v [Zhu96]. It is known that the only such form with odd weight is zero. (See, for example [Ser73].) Therefore only even n can give non-zero graded traces. Dong and Mason did the $n = 0$ case [DM00]. The next case of interest is $n = 2$.

Henceforth fix a non-zero $\lambda \in 2\Lambda$ and an orthonormal basis $\{\alpha_1, \alpha_2, \ldots, \alpha_{24}\}$ of $\mathfrak{h}$ such that $\lambda = \ell\alpha_1$. The subspace of $M(1)$ composed of weight-two elements has basis

$$\{\alpha_i(-1)^2, \alpha_i(-1)\alpha_j(-1), \alpha_i(-2) | i \neq j, 1 \leq i, j \leq 24\}.$$

Substituting the elements of this set for u in the expression $u \otimes e^\lambda + t(u) \otimes e^{-\lambda}$ produces a basis of $V(\lambda)_2^+$.

Using the Virasoro relations, $[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}c$, we can show that $L(1)\cdot v = 0$ and $L(2)\cdot v = 0$ imply that $L(n)\cdot v = 0$ for all $n \geq 3$, i.e., that v is a highest weight vector.

Direct computation, using $\omega = \frac{1}{2}\sum_{i=1}^{24}\alpha_i(-1)^2$ and

$$Y_{V^\natural}(\omega, z)v = Y_{V_\Lambda}(\omega, z)v = \sum_{n\in\mathbb{Z}} L(n)vz^{-n-2},$$

for $v \in V_\Lambda^+$, gives

$$\begin{aligned} L(1) &= \sum_{i=1}^{24}\sum_{n\geq 1}\alpha_i(1-n)\alpha_i(n), \\ L(2) &= \sum_{i=1}^{24}\Big(\frac{1}{2}\alpha_i(1)^2 + \sum_{n\geq 2}\alpha_i(2-n)\alpha_i(n)\Big). \end{aligned}$$

We know that $L(1)$ takes $V(\lambda)_2^+$ to $V(\lambda)_1^+$ and $L(2)$ takes $V(\lambda)_2^+$ to $V(\lambda)_0^+$. Since $V(\lambda)_1^+ \cap V(\lambda)_0^+ = 0$, we can define

$$\begin{gathered} L(1) + L(2) : V(\lambda)_2^+ \to V(\lambda)_1^+ \oplus V(\lambda)_0^+ \\ \big(L(1) + L(2)\big)v \mapsto L(1)v + L(2)v. \end{gathered}$$

The kernel of $L(1) + L(2)$ equals the intersection of the kernels of $L(1)$ and $L(2)$, which is the set of highest weight vectors in $V(\lambda)_2^+$.

Let $2 \leq i, j \leq 24$ with $i \neq j$. The actions of $L(1) + L(2)$ on our basis elements are

$$\begin{aligned}
\alpha_1(-1)^2 \otimes (e^{\lambda} + e^{-\lambda}) &\mapsto 2\ell\alpha_1(-1) \otimes (e^{\lambda} - e^{-\lambda}) + 1 \otimes (e^{\lambda} + e^{-\lambda}), \\
\alpha_i(-1)^2 \otimes (e^{\lambda} + e^{-\lambda}) &\mapsto 1 \otimes (e^{\lambda} + e^{-\lambda}), \\
\alpha_1(-1)\alpha_i(-1) \otimes (e^{\lambda} + e^{-\lambda}) &\mapsto \ell\alpha_i(-1) \otimes (e^{\lambda} - e^{-\lambda}), \\
\alpha_i(-1)\alpha_j(-1) \otimes (e^{\lambda} + e^{-\lambda}) &\mapsto 0, \\
\alpha_1(-2) \otimes (e^{\lambda} - e^{-\lambda}) &\mapsto 2\alpha_1(-1) \otimes (e^{\lambda} - e^{-\lambda}) + 2\ell \otimes (e^{\lambda} + e^{-\lambda}), \\
\alpha_i(-2) \otimes (e^{\lambda} - e^{-\lambda}) &\mapsto 2\alpha_i(-1) \otimes (e^{\lambda} - e^{-\lambda}).
\end{aligned}$$

Finding the null space of $L(1) + L(2)$ yields the following proposition.

Proposition 12.4. *The following set is a basis for the space of highest weight-vectors in* $V(\lambda)_2^+, \lambda \neq 0$.

$$\begin{aligned}
\big\{\alpha_i(-1)\alpha_j(-1) \otimes (e^{\lambda} + e^{-\lambda}), 2\alpha_1(-1)\alpha_i(-1) \otimes (e^{\lambda} + e^{-\lambda}) - \ell\alpha_i(-2) \otimes (e^{\lambda} - e^{-\lambda}), \\
- \ell\alpha_1(-2) \otimes (e^{\lambda} + e^{-\lambda}) + \left(2\ell^2 - 1\right) \alpha_i^2 \otimes (e^{\lambda} + e^{-\lambda}) + \alpha_1(-1)^2 \otimes (e^{\lambda} + e^{-\lambda}) \\
\big| 2 \leq i, j \leq 24, i \neq j\big\}
\end{aligned}$$

The first two types of vector listed in the proposition have graded trace equal to zero. That the last type does not is an implication of our main theorem.

13 Main Theorem

Recall that Λ is the Leech lattice, λ is a non-zero element of 2Λ and $\{\alpha_1, \alpha_2, \cdots, \alpha_{24}\}$ is an orthonormal basis of $\mathfrak{h}$ such that $\lambda = \ell\alpha_1$ for some $\ell \in \mathbb{C}$. Fix a basis element α_i, $i \neq 1$ and define the following element of $V^\natural$;

$$
\begin{aligned}
v(\lambda) := -\ell\left(\alpha_1(-2)\otimes(e^{\lambda}-e^{-\lambda})\right) \\
+ (2\ell^2-1)\left(\alpha_i(-1)^2\otimes(e^{\lambda}+e^{-\lambda})\right) + \alpha_1(-1)^2\otimes(e^{\lambda}+e^{-\lambda}). \quad (59)
\end{aligned}
$$

Theorem 13.1. *Let $0\neq\lambda\in 2\Lambda$ and let $v(\lambda)$ be the element of $V^\natural$ defined above. Then $v(\lambda)$ is a highest-weight vector of weight $k=(\lambda,\lambda)/2+2$ and*

$$
\begin{aligned}
Z_{V^\natural}(v(\lambda),\tau) = \frac{(\lambda,\lambda)}{8}\eta(\tau)^{12}\Bigg\{ 2\big(E_2(\tau)-2E_2(2\tau)\big)\left(\frac{\Theta_1(\tau)}{2}\right)^{(\lambda,\lambda)-12} \\
+\big(-E_2(\tau/2)+2E_2(\tau)\big)\left(\frac{\Theta_2(\tau)}{2}\right)^{(\lambda,\lambda)-12} \\
+\big(-E_2(\tau/2)+4E_2(\tau)-4E_2(2\tau)\big)\left(\frac{\Theta_3(\tau)}{2}\right)^{(\lambda,\lambda)-12}\Bigg\}.
\end{aligned}
$$

Remark 13.2. *If $\lambda=0$ or $\lambda\notin 2\Lambda$ then $v(\lambda)$ is still a highest-weight vector; however, in those cases $Z_{V^\natural}(v(\lambda),\tau)=0$. For $\lambda\in 2\Lambda$, if $(\lambda,\lambda)\geq 32$, then $Z_{V^\natural}(v(\lambda),z)$ is non-zero.*

Proposition 12.4 asserts that $v(\lambda)$ is a highest weight vector. Computing its weight using the $L(0)$ operator is straightforward. To compute the graded trace of $v(\lambda)$ on $V^\natural = V_\Lambda^+\oplus(V_\Lambda^T)^+$, we start with the trace on V_Λ^+.

By Proposition 12.3, to calculate $Z_{V_\Lambda^+}(v,q)$ we need only consider the trace of $o(v(\lambda))$ on $V(\beta)^+$, where $\beta=\lambda/2$.

Let $u(\beta)=u\otimes e^{\beta}+t(u)\otimes e^{-\beta}\in V(\beta)^+$ and

$$
v := -\ell\alpha_1(-2)+\left(2\ell^2-1\right)\alpha_i(-1)^2+\alpha_1(-1)^2. \quad (60)
$$

We can write $v(\lambda)$ as $v\otimes e^{\lambda}+t(v)\otimes e^{-\lambda}$. Only the terms of $Y(v(\lambda),z)u(\beta)$ in $V(\beta)^+$ can contribute to the trace. These are

$$
Y(v\otimes e^{\lambda},z)t(u)\otimes e^{-\beta}+Y(t(v)\otimes e^{-\lambda},z)u\otimes e^{\beta}.
$$

Let $E^-(\mp\lambda, z)E^+(\mp\lambda, z) = \sum_{n\in\mathbb{Z}} E^{\mp}(n)z^{-n}$. The vertex operators on V_Λ^+ are the restrictions of those on V_Λ given in Section 11. Bearing in mind that $2\beta = \lambda = \ell\alpha_1$, we calculate:

$$\begin{aligned}
&Y(\pm\alpha_1(-2)\otimes e^{\pm\lambda}, z)u\otimes e^{\mp\beta}\\
&\qquad = \pm {}^{\circ}_{\circ} \sum_{m,n\in\mathbb{Z}} (-m-1)\alpha_1(m)E^{\mp}(n)z^{-m-n-2}e^{\pm\lambda}z^{\pm\lambda} {}^{\circ}_{\circ}\, u\otimes e^{\mp\beta},\\
&\qquad = z^{-\ell^2/2} {}^{\circ}_{\circ} \Big((\mp\alpha_1, \mp\beta)z^{-2} \pm \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} (-m-1)\alpha_1(m)z^{-m-2}\Big)\\
&\qquad\quad \cdot \sum_{n\in\mathbb{Z}} E^{\mp}(n)z^{-n} {}^{\circ}_{\circ}\, u\otimes e^{\pm\beta},
\end{aligned}$$

$$\begin{aligned}
&Y(\alpha_k(-1)^2\otimes e^{\pm\lambda}, z)u\otimes e^{\mp\beta}\\
&\qquad = {}^{\circ}_{\circ} \sum_{j,m,n\in\mathbb{Z}} \alpha_k(j)\alpha_k(m)E^{\mp}(n)z^{-j-m-n-2}e^{\pm\lambda}z^{\pm\lambda} {}^{\circ}_{\circ}\, u\otimes e^{\mp\beta},\\
&\qquad = z^{-\ell^2/2} {}^{\circ}_{\circ} \Big((\alpha_k, \mp\beta)^2 z^{-2} + 2(\alpha_k, \mp\beta)\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \alpha_k(m)z^{-m-2}\\
&\qquad\quad + \sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} \alpha_k(j)\alpha_k(m)z^{-j-m-2}\Big) \sum_{n\in\mathbb{Z}} E^{\mp}(n)z^{-n} {}^{\circ}_{\circ}\, u\otimes e^{\pm\beta}.
\end{aligned}$$

We wish to calculate the trace of $o(v(\lambda)) = \mathrm{Res}_z\, z^{\ell^2/2+1}Y(v(\lambda), z)$. By equating the coefficients of $z^{-2-\ell^2/2}$ in the above equations, we have

$$\begin{aligned}
o\big(\pm\alpha_1(-2)\otimes e^{\pm\lambda}\big)u\otimes e^{\mp\beta} = \Big(&\frac{\ell}{2}E^{\mp}(0)u\\
&\pm \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} (-m-1) {}^{\circ}_{\circ}\, \alpha_1(m)E^{\mp}(-m) {}^{\circ}_{\circ}\, u\Big)\otimes e^{\pm\beta},
\end{aligned} \tag{61}$$

for $i \neq 1$,

$$o(\alpha_i(-1)^2\otimes e^{\pm\lambda})u\otimes e^{\mp\beta} = \Big(\sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^{\circ}_{\circ}\, \alpha_i(j)\alpha_i(m) {}^{\circ}_{\circ}\, E^{\mp}(-j-m)u\Big)\otimes e^{\pm\beta}, \tag{62}$$

$$o(\alpha_1(-1)^2\otimes e^{\pm\lambda})u\otimes e^{\mp\beta} = \Big(\frac{\ell^2}{4}E^{\mp}(0)u \mp \ell \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} {}^{\circ}_{\circ}\,\alpha_1(m)E^{\mp}(-m)\,{}^{\circ}_{\circ}\,u$$
$$+ \sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^{\circ}_{\circ}\,\alpha_1(j)\alpha_1(m)E^{\mp}(-j-m)\,{}^{\circ}_{\circ}\,u\Big)\otimes e^{\pm\beta}. \tag{63}$$

Combining equations (61), (62) and (63) as indicated by the definition of v, equation (60), gives

$$o(v\otimes e^{\lambda})u\otimes e^{-\beta} = \big(O^{+}u\big)\otimes e^{\beta},$$
$$o(t(v)\otimes e^{-\lambda})u\otimes e^{\beta} = \big(O^{-}u\big)\otimes e^{-\beta}.$$

where

$$O^{\pm} := -\frac{\ell^2}{4}E^{\mp}(0) \pm \ell \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} m\,{}^{\circ}_{\circ}\,\alpha_1(m)E^{\mp}(-m)\,{}^{\circ}_{\circ}$$
$$+ (2\ell^2-1)\sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^{\circ}_{\circ}\,\alpha_i(j)\alpha_i(m)\,{}^{\circ}_{\circ}\,E^{\mp}(-j-m) + \sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^{\circ}_{\circ}\,\alpha_1(j)\alpha_1(m)E^{\mp}(-j-m)\,{}^{\circ}_{\circ}\,. \tag{64}$$

To compute the trace of $o(v(\lambda))q^{L(0)}$ on $V(\beta)^{+}$ we use the decomposition

$$V(\beta)^{+} = M(1)^{+}\otimes\mathbb{C}\left(e^{\beta}+e^{-\beta}\right)\oplus M(1)^{-}\otimes\mathbb{C}\left(e^{\beta}-e^{\beta}\right)$$

in conjunction with the following lemma.

Lemma 13.3.

$$\operatorname{tr} o(v(\lambda))q^{L(0)}\Big|_{M(1)^{+}\otimes\mathbb{C}(e^{\beta}+e^{-\beta})} = q^{\ell^2/8}\operatorname{tr} O^{\pm}q^{L(0)}\Big|_{M(1)^{+}}$$
$$\operatorname{tr} o(v(\lambda))q^{L(0)}\Big|_{M(1)^{-}\otimes\mathbb{C}(e^{\beta}-e^{-\beta})} = -q^{\ell^2/8}\operatorname{tr} O^{\pm}q^{L(0)}\Big|_{M(1)^{-}}$$

Proof. We prove the second equation. The proof of the first is very similar. Let $\{u_\gamma|\gamma\in C\}$ be a basis of $M(1)^{-}$, then $\{u_\gamma\otimes(e^{\beta}-e^{-\beta})|\gamma\in C\}$ is a basis of $M(1)^{-}\otimes\mathbb{C}(e^{\beta}-e^{-\beta})$. We apply

$o(v(\lambda))q^{L(0)}$ to $u_\alpha \otimes (e^\beta - e^{-\beta})$, project into $V(\beta)$ and get

$$\big(-\big(O^+u_\alpha\big)\otimes e^\beta + \big(O^-u_\alpha\big)\otimes e^{-\beta}\big)q^{\mathrm{wt}\,u_\alpha+\ell^2/8}. \tag{65}$$

Because t preserves $V(\beta)$, and $v(\lambda)$ and $u_\alpha \otimes (e^\beta - e^{-\beta})$ are in V_Λ^+, expression (65) is in $V(\beta)^+[q]$, in particular it is fixed by t. Thus

$$-t\big(O^+u_\alpha\big) = O^-u_\alpha. \tag{66}$$

Let $\{v_\delta | \delta \in D\}$ be a basis of $M(1)^+$, and write $O^+u_\alpha = \sum_{\gamma\in C} c_\gamma u_\gamma + \sum_{\delta\in D} c_\delta v_\delta$ and $O^-u_\alpha = \sum_{\gamma\in C} d_\gamma u_\gamma + \sum_{\delta\in D} d_\delta v_\delta$ for some $c_\gamma, c_\delta, d_\gamma, d_\delta \in \mathbb{C}$. From (66), $c_\gamma = d_\gamma$ and $-c_\delta = d_\delta$. We rewrite (65) as

$$q^{\ell^2/8}\Big(-\sum_{\gamma\in C} c_\gamma u_\gamma \otimes \big(e^\beta - e^{-\beta}\big) - \sum_{\delta\in D} c_\delta v_\delta \otimes \big(e^\beta + e^{-\beta}\big)\Big)q^{\mathrm{wt}\,u_\alpha}.$$

The projection into the span of $u_\alpha \otimes (e^\beta - e^{-\beta})$ is $-q^{\ell^2/8}c_\alpha q^{\mathrm{wt}\,u_\alpha}$; thus

$$\mathrm{tr}\, o(v(\lambda))q^{L(0)}\Big|_{M(1)^-\otimes\mathbb{C}(e^\beta - e^{-\beta})} = -q^{\ell^2/8}\sum_{\gamma\in C} c_\gamma q^{\mathrm{wt}\,u_\gamma}.$$

Clearly $\mathrm{tr}\, O^\pm q^{L(0)}\big|_{M(1)^-} = \sum_{\gamma\in C} c_\gamma q^{\mathrm{wt}\,u_\gamma}$ and the result follows. □

Hence it remains to compute the trace of $O^\pm q^{L(0)}$ on $M(1)^\pm$. To do this we define $x^N \in (\mathrm{End}\, M(1))\,[x]$ by

$$x^N\left(\alpha_1(-n_1)\alpha_2(-n_2)\ldots\alpha_k(-n_k)\right) = x^k\alpha_1(-n_1)\alpha_2(-n_2)\ldots\alpha_k(-n_k), \tag{67}$$

for $\alpha_i \in \mathfrak{h}$ and $n_i > 0$ and compute the trace of $O^\pm x^n q^{L(0)}$ on $M(1)$.

Lemma 13.4.

$$\mathrm{tr}\, O^\pm x^N q^{L(0)}\Big|_{M(1)} \qquad = \qquad \frac{\ell^2}{4}\Big(-1 \quad + \quad 24\sum_{n>0}\frac{nxq^n}{1-xq^n}\Big)\frac{\exp\Big(\sum_{n>0}\frac{-\ell^2xq^n}{n(1-xq^n)}\Big)}{\prod_{n>0}(1-xq^n)^{24}}$$

Proof. Dong and Mason [DM00, Lemma 4.2] show that

$$\operatorname{tr} E^{\pm}(0)x^N q^{L(0)}\Big|_{M(1)} = \frac{\exp\left(\sum_{n>0}\frac{-\ell^2 x q^n}{n(1-xq^n)}\right)}{\prod_{n>0}(1-xq^n)^{24}}.$$

We compute the traces of the remaining components of $O^{\pm}$, from equation (64), using their methodology.

Let $A = \mathbb{C}\alpha_1$, $B = \mathbb{C}\alpha_i$ for $i \neq 1$, and $\mathfrak{h} = A \oplus B \oplus C$ be an orthogonal direct sum, then

$$M(1) = S(\hat{\mathfrak{h}}^-) = S(\hat{A}^-) \otimes S(\hat{B}^-) \otimes S(\hat{C}^-).$$

Furthermore, if T, U, V and W are linear endomorphisms such that $T|_{M(1)} = U|_{S(\hat{A}^-)} \otimes V|_{S(\hat{B}^-)} \otimes W|_{S(\hat{C}^-)}$, then

$$\operatorname{tr} T = \operatorname{tr} U \cdot \operatorname{tr} V \cdot \operatorname{tr} W.$$

It is not hard to see that

$$\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} m {}^\circ_\circ \alpha_1(m)E^{\mp}(-m) {}^\circ_\circ x^N q^{L(0)}\Big|_{M(1)}$$

$$= \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} m {}^\circ_\circ \alpha_1(m)E^{\mp}(-m) {}^\circ_\circ x^N q^{L(0)}\Big|_{S(\hat{A}^-)} \otimes\ x^N q^{L(0)}\Big|_{S(\widehat{(B\oplus C)}^-)},$$

$$\sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^\circ_\circ \alpha_1(j)\alpha_1(m)E^{\mp}(-j-m) {}^\circ_\circ x^N q^{L(0)}\Big|_{M(1)}$$

$$= \sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^\circ_\circ \alpha_1(j)\alpha_1(m)E^{\mp}(-j-m) {}^\circ_\circ x^N q^{L(0)}\Big|_{S(\hat{A}^-)} \otimes\ x^N q^{L(0)}\Big|_{S(\widehat{(B\oplus C)}^-)}.$$

For $\sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^\circ_\circ \alpha_i(j)\alpha_i(m) {}^\circ_\circ E^{\mp}(-j-m)$, write a generic element of $M(1)$ as pq, where $p \in S(\hat{B}^-)$ and $q \in S(\widehat{(A\oplus C)}^-)$. Here p is a polynomial in $\alpha_i(-n)$'s $n \in \mathbb{N}$, of degree k. Assume that ${}^\circ_\circ \alpha_i(m)\alpha_i(j) {}^\circ_\circ E^{\mp}(-m-j)$ has a non-zero contribution to the trace on $M(1)$. It must preserve the degree of p. The terms of $E^{\mp}(-m-j)$ have no effect on the degree of p, because $(\lambda, \alpha_i) = 0$, while $\alpha_i(m)\alpha_i(j)$ takes p to 0 or a polynomial of degree $k - (m+j)$. Thus $m + j = 0$, and the only terms of $\sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^{\circ}_{\circ}\, \alpha_i(j)\alpha_i(m) \,{}^{\circ}_{\circ}\, E^{\mp}(-j-m)pq$ that contribute to the trace are $\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} {}^{\circ}_{\circ}\, \alpha_i(m)\alpha_i(-m) \,{}^{\circ}_{\circ}\, E^{\mp}(0)pq$. Furthermore,

$$\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} {}^{\circ}_{\circ}\, \alpha_i(m)\alpha_i(-m) \,{}^{\circ}_{\circ}\, E^{\mp}(0)x^N q^{L(0)}\Big|_{M(1)} = E^{\mp}(0)x^N q^{L(0)}\Big|_{S(\hat{A}^-)}$$

$$\otimes\ 2\sum_{m\in\mathbb{N}} \alpha_i(-m)\alpha_i(m)x^N q^{L(0)}\Big|_{S(\hat{B}^-)} \otimes\ x^N q^{L(0)}\Big|_{S(\hat{C}^-)}.$$

We know that for any vector space W of dimension k

$$\operatorname{tr} x^N q^{L(0)}\Big|_{S(\hat{W}^-)} = \prod_{n>0} \frac{1}{(1-xq^n)^k}.$$

Additionally, Dong and Mason [DM00, Lemmma 4.1] show that

$$\operatorname{tr} E^{\pm}(0)x^N q^{L(0)}\Big|_{S(\hat{A}^-)} = \frac{\exp\left(\sum_{n>0} \frac{-\ell^2 xq^n}{n(1-xq^n)}\right)}{\prod_{n>0}(1-xq^n)}.$$

The remaining traces necessary to find $\operatorname{tr} O^{\pm}x^N q^{L(0)}\big|_{M(1)}$ are given in the following lemma.

Lemma 13.5. *We have*

$$\operatorname{tr} \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} m \,{}^{\circ}_{\circ}\, \alpha_1(m)E^{\mp}(-m) \,{}^{\circ}_{\circ}\, x^N q^{L(0)}\Big|_{S(\hat{A}^-)} =$$

$$\pm 2\ell \sum_{n>0} \frac{nxq^n}{1-xq^n} \frac{\exp\left(\sum_{n>0} \frac{-\ell^2 xq^n}{n(1-xq^n)}\right)}{\prod_{n>0} 1-xq^n}, \quad (68)$$

$$\operatorname{tr} \sum_{m\in\mathbb{N}} \alpha_i(-m)\alpha_i(m)x^N q^{L(0)}\Big|_{S(\hat{B}^-)} = \sum_{n>0} \frac{nxq^n}{1-xq^n} \prod_{n>0} \frac{1}{1-xq^n}, \quad (69)$$

$$\operatorname{tr} \sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^{\circ}_{\circ}\, \alpha_1(j)\alpha_1(m)E^{\mp}(-j-m)\, {}^{\circ}_{\circ}\, x^N q^{L(0)}\Big|_{S(\hat{A}^-)} =$$

$$2\sum_{n>0} \frac{nxq^n}{1-xq^n} \frac{\exp\left(\sum_{n>0} \frac{-\ell^2 xq^n}{n(1-xq^n)}\right)}{\prod_{n>0} 1-xq^n}. \qquad (70)$$

Proof. We prove equation (68) in some detail and sketch proofs of equations (69) and (70)

The set $\{\alpha_1(-n)^{k_n}\alpha_1(-(n-1))^{k_{n-1}}\cdots\alpha_1(-1)^{k_1} | k_i \geq 0, n \geq 1\}$ is a basis for $S(\hat{A}^-)$. Fix a basis element $\boldsymbol{\alpha} = \alpha_1(-n)^{k_n}\alpha_1(-(n-1))^{k_{n-1}}\cdots\alpha_1(-1)^{k_1}$, and for convenience set $m\, {}^{\circ}_{\circ}\, \alpha_1(m)E^{\mp}(-m)\, {}^{\circ}_{\circ} = f^{\mp}(m)$. We find the projection, $P_{\boldsymbol{\alpha}}$ into the space $\mathbb{C}\boldsymbol{\alpha}$ of $\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} f^{\mp}(m)\boldsymbol{\alpha}$. Only those summands that preserve the degree of all of the $\alpha_1(-j)$ in $\boldsymbol{\alpha}$ will have non-zero contributions to this projection. For $\sum_{m>0} f^{\mp}(m)$ these are

$$\sum_{m=1}^{n} \sum_{\substack{p_j=0\\ 0\leq j\leq n\\ j\neq m}}^{k_j} \sum_{p_m=0}^{k_m-1} m \frac{\big(\pm\lambda(-1)\big)^{p_1}}{p_1!} \cdots \frac{\big(\pm\lambda(-m)\big)^{p_m+1}}{m^{p_m+1}(p_m+1)!} \cdots$$

$$\frac{\big(\pm\lambda(-n)\big)^{p_n}}{n^{p_n}p_n!} \frac{\big(\mp\lambda(1)\big)^{p_1}}{p_1!} \cdots \frac{\alpha_1(m)\big(\mp\lambda(m)\big)^{p_m}}{m^{p_m}p_m!} \cdots \frac{\big(\mp\lambda(n)\big)^{p_n}}{n^{p_n}p_n!}. \qquad (71)$$

Using equation (71) and the actions of the $\alpha_1(n)$'s given in equation (55), we calculate:

$$P_{\boldsymbol{\alpha}}\Big(\sum_{m>0} f^{\mp}(m)\boldsymbol{\alpha}\Big) = \pm \sum_{m=1}^{n} \sum_{\substack{p_j=0\\ 1\leq j\leq n}}^{k_j} (-1)^{p_1+\cdots+p_m} m$$

$$\cdot \frac{\ell^{2p_m+1} m^{p_m+1} k_m \cdots (k_m-p_m)}{m^{2p_m+1}(p_m+1)!p_m!} \prod_{\substack{j=1\\ j\neq m}}^{n} \frac{\ell^{2p_j} j^{p_j} k_j \cdots (k_j-p_j+1)}{j^{2p_j}(p_j!)^2} \boldsymbol{\alpha},$$

$$= \pm\ell \sum_{m=1}^{n} \sum_{\substack{p_j=0\\ 1\leq j\leq n}}^{k_j} m \frac{\binom{k_m}{p_m+1}}{p_m!} \left(\frac{-\ell^2}{m}\right)^{p_m} \prod_{\substack{j=1\\ j\neq m}}^{n} \frac{\binom{k_j}{p_j}}{p_j!} \left(\frac{-\ell^2}{j}\right)^{p_j} \boldsymbol{\alpha}.$$

A similar calculation for $m > 0$ shows that $P_{\boldsymbol{\alpha}}(\sum_{m<0} f^{\mp}(m)\boldsymbol{\alpha})$ is the same as $P_{\boldsymbol{\alpha}}(\sum_{m>0} f^{\mp}(m)\boldsymbol{\alpha})$. So $P_{\boldsymbol{\alpha}}(\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} f^{\mp}(m)\boldsymbol{\alpha}) = 2P_{\boldsymbol{\alpha}}(\sum_{m>0} f^{\mp}(m)\boldsymbol{\alpha})$.

To compute the trace we sum the above coefficient of $\boldsymbol{\alpha}$ over all possible choices of $\boldsymbol{\alpha}$, that is over all n's and k_j's. Let $\mathbb{Z}^\infty = \bigoplus_{j\geq 1}\mathbb{Z}$, i.e., the space of $\mathbf{n} = (n_1, n_2, \ldots, n_j, \ldots)$ such that $n_j \in \mathbb{Z}$ and $n_j = 0$ for all but finitely many j. Put a partial ordering on $\mathbb{Z}^\infty$ such that $\mathbf{n} \leq \mathbf{m}$, if $n_j \leq m_j$ for all j. Then

$$\operatorname{tr} \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} m \,{}^\circ_\circ\, \alpha_1(m)E^{\mp}(-m) \,{}^\circ_\circ\, x^N q^{L(0)}\Big|_{S(\hat{A}^-)} = \pm 2\ell \sum_{m>0} m \sum_{\substack{\mathbf{k}\in\mathbb{Z}^\infty\\ \mathbf{k}\geq\mathbf{0}}} \sum_{\substack{\mathbf{p}\in\mathbb{Z}^\infty\\ \mathbf{0}\leq\mathbf{p}\leq\mathbf{k}}} \frac{\binom{k_m}{p_m+1}}{p_m!}$$

$$\cdot \left(\frac{-\ell^2}{m}\right)^{p_m} \prod_{\substack{j\geq 1\\ j\neq m}} \frac{\binom{k_j}{p_j}}{p_j!}\left(\frac{-\ell^2}{j}\right)^{p_j} x^{\sum_{j\geq 0} k_j} q^{\sum_{j\geq 0} jk_j},$$

which equals

$$\pm 2\ell \sum_{m>0} m \Bigg(\sum_{k_m, p_m \geq 0} \frac{\binom{k_m}{p_m+1}}{p_m!}\left(\frac{-\ell^2}{m}\right)^{p_m} x^{k_m} q^{mk_m}\Bigg)$$

$$\cdot \prod_{\substack{k_j,p_j\geq 0\\ j\geq 1\\ j\neq m}} \Bigg(\sum_{k_j,p_j\geq 0} \frac{\binom{k_j}{p_j}}{p_j!}\left(\frac{-\ell^2}{j}\right)^{p_j} x^{k_j} q^{jk_j}\Bigg).$$

We can now rearrange this sum as Dong and Mason do in the proof of their Lemma 4.1 [DM00] and get equation (68).

We now turn to equation (69). Fix a basis element $\boldsymbol{\alpha} = \alpha_i(-n)^{k_n}\alpha_i(-(n-1))^{k_{n-1}}\cdots\alpha_i(-1)_1^k$ of $S(\hat{B}^-)$.

$$\sum_{m>0}\alpha_i(-m)\alpha_i(m)\boldsymbol{\alpha} = \sum_{m>0} mk_m\boldsymbol{\alpha}$$

Add over all basis elements to compute the trace;

$$\operatorname{tr}\sum_{m>0}\alpha_i(-m)\alpha_i(m)x^N q^{L(0)}\Big|_{S(\hat{B}^-)} = \sum_{\substack{\mathbf{k}\in\mathbb{Z}^\infty\\ \mathbf{k}\geq\mathbf{0}}} \sum_{m>0} mk_m \prod_{j>0} x^{k_j}q^{jk_j}.$$

We have

$$\sum_{k_m>0} mk_m x^{k_m} q^{mk_m} = q\frac{\partial}{\partial q}\left(\frac{1}{1-xq^m}\right) = \frac{mxq^m}{(1-xq^m)^2}.$$

So summing over the k_j's in $\mathbf{k}$ we conclude that

$$\operatorname{tr}\sum_{m>0}\alpha_i(-m)\alpha_i(m)x^N q^{L(0)}\Big|_{S(\hat{B}^-)} = \sum_{m>0}\frac{mxq^m}{1-xq^m}\prod_{j>0}\frac{1}{1-xq^j}.$$

To prove equation (70) write $\sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^\circ_\circ\,\alpha_1(j)\alpha_1(m)E^{\mp}(-j-m)\,{}^\circ_\circ$ as

$$\sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0\\ j\neq\pm m}} {}^\circ_\circ\,\alpha_1(j)\alpha_1(m)E^{\mp}(-j-m)\,{}^\circ_\circ$$
$$+\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} {}^\circ_\circ\,\alpha_1(m)^2E^{\mp}(-2m)\,{}^\circ_\circ + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} {}^\circ_\circ\,\alpha_1(-m)\alpha_1(m)E^{\mp}(0)\,{}^\circ_\circ\,.$$

The $m\neq\pm j$ terms make no contribution to the trace because each pair (m,j) with $j>0$ is cancelled by $(m,-j)$.

Calculating the second and third sums as in the proof of equation (68), we find:

$$\begin{aligned}
&\operatorname{tr}\sum_{\substack{j,m\in\mathbb{Z}\\ j,m\neq 0}} {}^\circ_\circ\,\alpha_1(j)\alpha_1(m)E^{\mp}(-j-m)\,{}^\circ_\circ\, x^N q^{L(0)}\Big|_{S(\hat{A}^-)} = \\
&\qquad 2\sum_{m>0}\frac{xq^m}{(1-xq^m)^2}\sum_{p_m\geq 0}\Big(\ell^2\frac{xq^m}{1-xq^m}+m(p_m+1)\Big)\frac{1}{p_m!}\left(\frac{-\ell^2xq^m}{m(1-xq^m)}\right)^{p_m}\\
&\qquad \cdot\prod_{\substack{j>0\\ j\neq m}}\frac{1}{1-xq^j}\sum_{p_j\geq 0}\frac{1}{p_j!}\Big(\frac{-\ell^2xq^j}{j(1-xq^j)}\Big)^{p_j}\\
&= \quad 2\sum_{m>0}\frac{mxq^m}{1-xq^m}\prod_{j>0}\frac{1}{1-xq^j}\sum_{p_j>0}\frac{1}{p_j!}\Big(\frac{-\ell^2xq^n}{n(1-xq^n)}\Big)^{p_j}.
\end{aligned}$$

The last equality follows from the identity:

$$m\sum_{p_m\geq 0}\frac{p_m}{p_m!}\left(\frac{-\ell^2xq^m}{m(1-xq^m)}\right)^{p_m} = -\ell^2\frac{xq^m}{1-xq^m}\sum_{p_m\geq 0}\frac{1}{p_m!}\left(\frac{-\ell^2xq^m}{m(1-xq^m)}\right)^{p_m},$$

and equation (70) now follows easily. □

This also completes the proof of Lemma 13.4. $\square$

We introduce the notation

$$f(q,x) := \frac{\exp\left(\sum_{n>0} \frac{-\ell^2 x q^n}{n(1-xq^n)}\right)}{\prod_{n>0}(1-xq^n)^{24}}, \qquad T(q,x) := \sum_{n>0} \frac{nxq^n}{1-xq^n},$$

and rewrite Lemma 13.4 in terms of this notation;

$$\operatorname{tr} O^{\pm} x^N q^{L(0)}\Big|_{M(1)} = \frac{\ell^2}{4}\big(-1+24T(q,x)\big)f(q,x).$$

It is not hard to see that

$$\operatorname{tr} O^{\pm} q^{L(0)}\Big|_{M(1)^+} = \frac{\ell^2}{8}\Big(\big(-1+24T(q,1)\big)f(q,1) + \big(-1+24T(q,-1)\big)f(q,-1)\Big), \tag{72}$$

$$\operatorname{tr} O^{\pm} q^{L(0)}\Big|_{M(1)^-} = \frac{\ell^2}{8}\Big(\big(-1+24T(q,1)\big)f(q,1) - \big(-1+24T(q,-1)\big)f(q,-1)\Big). \tag{73}$$

Recall that the trace of $o(v(\lambda))$ on V_Λ^+ equals the trace on $V(\beta)^+$ which equals the sum of the traces on $M(1)^+ \otimes (e^\beta + e^{-\beta})$ and $M(1)^- \otimes (e^\beta - e^{-\beta})$. From Lemma 13.3 and equations (72) and (73), we have

$$\operatorname{tr} o(v(\lambda)) q^{L(0)}\Big|_{V_\Lambda^+} = q^{\ell^2/8}\frac{\ell^2}{4}\big(-1+24T(q,-1)\big)f(q,-1) \tag{74}$$

From Lemmas 4.3 and 4.4 of Dong and Mason [DM00]

$$q^{\frac{\ell^2}{8}-1} f(q,-1) = \eta(\tau)^{12}\left(\frac{\Theta_1(\tau)}{2}\right)^{\ell^2-12}. \tag{75}$$

To rewrite $-1+24T(q,-1)$, we multiply the each term in the sum $T(q,-1) = -\sum_{n>0}\frac{nq^n}{1+q^n}$ by $\frac{1-q^n}{1-q^n}$, add and subtract nq^{2n} from the numerator and split the result into two sums to obtain

$$T(q,-1) = -\sum_{n>0}\frac{nq^n}{1-q^n} + 2\sum_{n>0}\frac{nq^{2n}}{1-q^{2n}}.$$

The function $E_2(\tau) := 1 - 24\sum_{n>0} \sigma_1(n) q^n = 1 - 24 \sum_{n>0} \frac{nq^n}{1-q^n}$. So

$$-1 + 24T(q,-1) = E_2(\tau) - 2E_2(2\tau). \tag{76}$$

The contribution of V_Λ^+ to $Z_{V^\natural}(v,\tau)$ is $q^{-1}\operatorname{tr} o(v) q^{L(0)}|_{V_\Lambda^+}$, and equations (74), (75) and (76) yield the following lemma.

Lemma 13.6. *The contribution of V_Λ^+ to $Z_{V^\natural}(v(\lambda),\tau)$ is*

$$\frac{\ell^2}{8}\eta(\tau)^{12} 2\big(E_2(\tau) - 2E_2(2\tau)\big)\left(\frac{\Theta_1(\tau)}{2}\right)^{\ell^2-12}.$$

Recall that $V^\natural = V_\Lambda^+ \oplus (V_\Lambda^T)^+$. We must now compute the contribution of $(V_\Lambda^T)^+$ to the graded trace of $v(\lambda)$. A twisted module for V_Λ, V_Λ^T is the tensor product of two spaces, $S(\mathfrak{h}[-1]^-)$ and T. The first, $S(\mathfrak{h}[-1]^-)$, is quite similar to $M(1)$, described in Section 11. It is a module for $\mathfrak{h}[-1] := \mathfrak{h} \otimes t^{\frac{1}{2}}\mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c$ induced from the one-dimensional module for the non-negatively graded subalgebra of $\mathfrak{h}[-1]$, on which c acts as the identity and the positively graded subalgebra acts as zero. The second part, T, is the projective 2^{12}-dimensional representation for Λ such that 2Λ acts trivially. The involution t acts on $S(\mathfrak{h}[-1]^-)$ in the same way it does on the $M(1)$ in V_Λ (see equation (57)) and on T as multiplication by -1. Thus the elements of V_Λ^T fixed by t are

$$\left(V_\Lambda^T\right)^+ = S(\mathfrak{h}[-1]^-)^- \otimes T.$$

The action of V_Λ^+ on $\left(V_\Lambda^T\right)^+$ is given by the twisted vertex operator

$$Y_{\mathbb{Z}+\frac{1}{2}}(v,z) := Y_0(\exp(\Delta_z)v, z).$$

Here for $w = \alpha_1(-n_1)\cdots\alpha_k(-n_k)\otimes e^{\gamma} \in V_\Lambda$;

$$\begin{aligned}
Y_0(w,z) &:= {}^{\circ}_{\circ}\, \frac{1}{(n_1-1)!}\left(\frac{\partial}{\partial z}\right)^{n_1-1}\alpha_{\frac{1}{2},1}(z)\cdots\frac{1}{(n_k-1)!}\left(\frac{\partial}{\partial z}\right)^{n_k-1}\alpha_{\frac{1}{2},k}(z) \\
&\qquad\qquad \cdot 2^{-(\gamma,\gamma)}E^{-}_{\frac{1}{2}}(-\gamma,z)E^{+}_{\frac{1}{2}}(-\gamma,z)\gamma z^{\frac{-(\gamma,\gamma)}{2}}\, {}^{\circ}_{\circ}, \\
\alpha_{\frac{1}{2}}(z) &:= \sum_{n\in\mathbb{Z}+\frac{1}{2}} \alpha(n)z^{-n-1}, \\
E^{\pm}_{\frac{1}{2}}(\alpha,z) &:= \exp\left(\sum_{n>0}\frac{\alpha\left(\pm(n+\frac{1}{2})\right)}{\pm(n+\frac{1}{2})}z^{\mp(n+\frac{1}{2})}\right).
\end{aligned}$$

Each $\alpha(n)\in\mathfrak{h}[-1]$ acts on $S(\mathfrak{h}[-1]^-)$ and γ acts on T. The linear map $\exp(\Delta_z) : V_\Lambda \to V_\Lambda\{z\}$ is defined in Frenkel, Lepowsky and Meurman [FLM88, Section 9.2].

The space $(V_\Lambda^T)^+$ is integrally graded by defining the weight of $\alpha_1(-n_1)\cdots\alpha_k(-n_k)\otimes t$ to be $\frac{3}{2}+\sum_{i=1}^k n_i$. The contribution of the twisted space to the graded trace of the element $v(\lambda)$ on $V^\natural$ is

$$q^{-1}\sum_{n\geq 1}\operatorname{tr}\big|_{(V_\Lambda^T)^+_n} o_{\frac{1}{2}}(v(\lambda))q^n,$$

where $o_{\frac{1}{2}}(v) := \operatorname{Res}_z z^{\operatorname{wt} v-1}Y_{\mathbb{Z}+\frac{1}{2}}(v,z)$.

Recall that $v(\lambda) = v\otimes e^{\lambda} + t(v)\otimes e^{-\lambda}$, where $v = -\ell\alpha_1(-2) + (2\ell^2+1)\alpha_i(-1)^2 + \alpha_1(-1)^2$. Clearly,

$$Y_{\mathbb{Z}+\frac{1}{2}}(v(\lambda),z) = Y_0(\exp(\Delta_z)v\otimes e^{\lambda},z) + Y_0(\exp(\Delta_z)t(v)\otimes e^{-\lambda},z).$$

With respect to our orthonormal basis of $\mathfrak{h}$, $\{\alpha_1,\alpha_2,\cdots\alpha_{24}\}$, Frenkel, Lepowsky and Meurman's equation for $\exp(\Delta_z)$ [FLM88, eq. 9.2.42] becomes

$$\begin{aligned}
\exp(\Delta_z) = 1 - \frac{1}{2}\sum_{k=1}^{24}\alpha_k(1)\alpha_k(0)z^{-1} + \frac{1}{16}\sum_{k=1}^{24}&\left(3\alpha_k(2)\alpha_k(0)+\alpha_k(1)^2\right)z^{-2} \\
&+ \frac{1}{8}\left(\sum_{k=1}^{24}\alpha_k(1)\alpha_k(0)\right)^2 z^{-2} + \cdots,
\end{aligned}$$

and we calculate

$$\begin{aligned}\exp(\Delta_z)v\otimes e^{\lambda}&=\Big(v-\ell\alpha_1(-1)z^{-1}+\frac{\ell^2}{8}z^{-2}\Big)\otimes e^{\lambda}\\ \exp(\Delta_z)t(v)\otimes e^{-\lambda}&=\Big(t(v)+\ell\alpha_1(-1)z^{-1}+\frac{\ell^2}{8}z^{-2}\Big)\otimes e^{-\lambda}.\end{aligned}\tag{77}$$

In analogy to the untwisted case, define $E^{\pm}_{\frac{1}{2}}(m)$ by

$$E^{-}_{\frac{1}{2}}(\pm\lambda,z)E_{\frac{1}{2}}(\pm\lambda,z)=\sum_{n\in\frac{1}{2}\mathbb{Z}}E^{\pm}_{\frac{1}{2}}(n)z^{-n}.$$

Since $\pm\lambda$ is in 2Λ, it acts trivially on V_Λ^T. Thus, using the definitions of $o_{\frac{1}{2}}$ and Y_0 and equation (77), we can show that

$$o_{\frac{1}{2}}(v(\lambda))=2^{-\ell^2}\big(O^{+}_{\frac{1}{2}}+O^{-}_{\frac{1}{2}}\big),\tag{78}$$

where

$$\begin{aligned}O^{\pm}_{\frac{1}{2}}=\mp\ell\sum_{m\in\mathbb{Z}+\frac{1}{2}}(-m-1)\alpha_1(m)E^{\mp}_{\frac{1}{2}}(-m)+(2\ell^2-1)\sum_{j,m\in\mathbb{Z}+\frac{1}{2}}\alpha_i(j)\alpha_i(m)E^{\mp}_{\frac{1}{2}}(-j-m)\\ +\sum_{j,m\in\mathbb{Z}+\frac{1}{2}}\alpha_1(j)\alpha_1(m)E^{\mp}_{\frac{1}{2}}(-j-m)\mp\ell\sum_{m\in\mathbb{Z}+\frac{1}{2}}\alpha_1(m)E^{\mp}_{\frac{1}{2}}(-m)+\frac{\ell^2}{8}E^{\mp}_{\frac{1}{2}}(0).\end{aligned}$$

Again we introduce the operator $x^N\in\operatorname{End}S\big(\hat{\mathfrak{h}}[-1]^-\big)[x]$ (see equation (67)). Using the same methods used to prove Lemma 13.4, we can prove the following lemma.

Lemma 13.7.

$$\operatorname{tr}O^{\pm}_{\frac{1}{2}}x^Nq^{L(0)}\Big|_{S(\hat{\mathfrak{h}}[-1]^-)}=\frac{\ell^2}{8}\big(1+24S(q,x)\big)g(q,x),$$

where $g(q,x)$ is the function used by Dong and Mason [DM00]

$$g(q,x):=q^{3/2}\frac{\exp\Big(\sum_{n>0}\frac{-\ell^2xq^{n+1/2}}{(n+\frac{1}{2})(1-xq^{n+1/2})}\Big)}{\prod_{n>0}\big(1-xq^{n+1/2}\big)^{24}},$$

and

$$S(q,x) := \sum_{n>0} (2n+1) \frac{xq^{n+1/2}}{1-xq^{n+1/2}}.$$

The trace of $O^{\pm}_{\frac{1}{2}} q^{L(0)}$ on $S(\hat{\mathfrak{h}}[-1]^-)^-$ is the difference of the trace of $O^{\pm}_{\frac{1}{2}} x^n q^{L(0)}$ on $S(\hat{\mathfrak{h}}[-1]^-)$ evaluated at $x = 1$ and the same trace evaluated at $x = -1$. The operator $o_{\frac{1}{2}}(v(\lambda))q^{L(0)}$ acts as the identity on T, thus it contributes 2^{12}, the dimension of T, to the trace. So from equation (78),

$$\begin{aligned} \operatorname{tr} o_{\frac{1}{2}}(v(\lambda))q^{L(0)}\Big|_{(V_\Lambda^T)^+} = 2^{12-\ell^2} \frac{\ell^2}{8} \Big(\big(1 + 24S(q,1)\big) g(q,1) \\ - \big(1 + 24S(q,-1)\big) g(q,-1) \Big). \end{aligned}$$

Dong and Mason show that

$$\begin{aligned} 2^{12-\ell^2} q^{-1} g(q,1) &= \eta(\tau)^{12} \left(\frac{\Theta_2(\tau)}{2} \right)^{\ell^2-12}, \\ 2^{12-\ell^2} q^{-1} g(q,-1) &= \eta(\tau)^{12} \left(\frac{\Theta_3(\tau)}{2} \right)^{\ell^2-12}. \end{aligned}$$

Recall that $T(q,x) = \sum_{n>0} \frac{nxq^n}{1-xq^n}$. By splitting the summation for $T(q^{1/2},x)$ into two sums according to the parity of the index variable, we see that

$$S(q,x) = T(q^{1/2},x) - 2T(q,x).$$

Use this identity and $E_2(\tau) = 1 - 24\sum_{n>0} \frac{nq^n}{1-q^n}$ to obtain

$$1 + 24S(q,1) = -E_2(\tau/2) + E_2(\tau).$$

Using the same identities and equation (76), we can get

$$1 + 24S(q,-1) = E_2(\tau/2) - 4E_2(\tau) + 4E_2(2\tau).$$

Put the above equations together to obtain the following lemma.

Lemma 13.8. *The contribution of* $\big(V_\Lambda^T\big)^+$ *to* $Z_{V^\natural}(v(\lambda),\tau)$ *is*

$$\frac{\ell^2}{8}\eta(\tau)^{12}\Big\{\big(-E_2(\tau/2)+2E_2(\tau)\big)\left(\frac{\Theta_2(\tau)}{2}\right)^{\ell^2-12} \\ +\big(-E_2(\tau/2)+4E_2(\tau)-4E_2(2\tau)\big)\left(\frac{\Theta_3(\tau)}{2}\right)^{\ell^2-12}\Big\}.$$

The main theorem now follows from Lemmas 13.6 and 13.8.

In general, the graded trace on $V^\natural$ of a highest-weight vector of positive weight is a cusp form [DM00, Proposition 2]. The Leech lattice is even and has no elements of square length two, so for $\lambda \in 2\Lambda$, $\ell^2 = (\lambda,\lambda)$ is divisible by eight and at least 16. This means that $\mathrm{wt}(v(\lambda)) = (\lambda,\lambda)/2+2$ is congruent to two modulo four and at least ten. There are no non-zero cusp forms of weights 10 or 14, so the main theorem for $(\lambda,\lambda) = 16, 24$ gives the identity;

$$2\big(E_2(\tau)-2E_2(2\tau)\big)\Theta_1(\tau)^n+\big(-E_2(\tau/2)+2E_2(\tau)\big)\Theta_2(\tau)^n \\ +\big(-E_2(\tau/2)+4E_2(\tau)-4E_2(2\tau)\big)\Theta_3(\tau)^n = 0 \quad (79)$$

where $n = 4$ or 12.

For $v(\lambda)$ of weight at least 18, i.e., $(\lambda,\lambda) \geq 32$, it is straightforward to show that $Z_{V^\natural}(v(\lambda),\tau)$ is non-zero by computing the coefficient of q. Combining our results with those of Dong and Mason [DM00, Theorem 3], there are $V^\natural$ highest-weight vectors of weights 12 and $2k$ for $k \geq 8$ with non-zero graded trace. The spaces of cusp forms of weights 12, 16, 18, 20, 22 and 26 are one-dimensional; so, for these weights, all cusp forms are achieved as the graded trace of a highest-weight vector.

Appendix A: Root systems

The following table of root systems for the simple finite-dimensional Lie algebras is complied from Humphreys [Hum78] and Fulton and Harris [FH91]. The vectors e_i are orthonormal.

Table 3: Simple root systems

	Root system
A_ℓ	$\{\pm(e_i - e_j) \mid 1 \le i < j \le \ell + 1\}$
B_ℓ	$\{\pm e_i, \pm(e_i \pm e_j) \mid 1 \le i < j \le \ell\}$
C_ℓ	$\{\pm 2e_i, \pm(e_i \pm e_j) \mid 1 \le i < j \le \ell\}$
D_ℓ	$\{\pm(e_i \pm e_j) \mid 1 \le i < j \le \ell\}$
E_6	$\{\pm(e_i \pm e_j), \frac{1}{2}(\pm e_1 \pm e_2 \pm \cdots \pm e_5 \pm \sqrt{3}e_6) \mid 1 \le i < j \le 5,$ number of minus signs in second term is even$\}$
E_7	$\{\pm(e_i \pm e_j), \pm\frac{1}{2}(\pm e_1 \pm e_2 \pm \cdots \pm e_6 + \sqrt{2}e_7) \mid 1 \le i < j \le 6,$ number of minus signs inside second parentheses is odd$\}$
E_8	$\{\pm(e_i \pm e_j), \frac{1}{2}(\pm e_1 \pm e_2 \pm \cdots \pm e_8) \mid 1 \le i < j \le 8,$ number of minus signs in second term is even$\}$
F_4	$\{\pm e_i, \pm(e_i \pm e_j), \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \mid 1 \le i < j \le 4\}$
G_2	$\pm\{e_1 - e_2, e_1 - e_3, e_2 - e_3, 2e_1 - e_2 - e_3, 2e_2 - e_1 - e_3, 2e_3 - e_1 - e_2\}$

Appendix B: Restatement of Theorem 13.1

The following proposition, which simplifies the statement of Theorem 13.1, was suggested by Michael Tuite.

Proposition 13.9. *Let the theta functions* $\Theta_1(\tau)$*,* $\Theta_2(\tau)$*,* $\Theta_3(\tau)$ *and the "Eisenstein series"* $E_2(\tau)$ *be defined as in Section 11. Then*

$$2\big(E_2(\tau) - 2E_2(2\tau)\big) = -\Theta_2^4(\tau) - \Theta_3^4(\tau), \tag{80}$$

$$-E_2(\tau/2) + 2E_2(\tau) = \Theta_1^4(\tau) + \Theta_3^4(\tau), \tag{81}$$

$$-E_2(\tau/2) + 4E_2(\tau) - 4E_2(2\tau) = \Theta_1^4(\tau) - \Theta_2^4(\tau). \tag{82}$$

Proof. The series $E_2(\tau)$ satisfies the following functional equation for any $g \in PSL_2(\mathbb{Z})$,

$$g \circ E_2(\tau) := E_2(g\tau) := E_2\Big(\frac{a\tau + b}{c\tau + d}\Big) = E_2(\tau)(c\tau + d)^2 + \frac{12c}{2\pi i}(c\tau + d), \tag{83}$$

see [Lan76, ch. X, §5]. Let

$$\Gamma(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z}) \middle| a \equiv d \equiv 1 \mod 2, b \equiv c \equiv 0 \mod 2 \right\}.$$

For any g in $\Gamma(2)$, let $g' = \begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix}$. Because c is congruent to zero modulo two, $g' \in PSL_2(\mathbb{Z})$; and we compute $2g\tau = \frac{a2\tau + 2b}{(c/2)2\tau + d} = g'2\tau$. Thus

$$g \circ E_2(2\tau) = E_2(2g\tau) = E_2(g'2\tau) = E_2(2\tau)(c\tau + d)^2 + \frac{6c}{2\pi i}(c\tau + d),$$

So for any $g \in \Gamma(2)$,

$$g \circ (E_2(\tau) - 2E_2(2\tau)) = (c\tau + d)^2(E_2(\tau) - 2E_2(2\tau)).$$

Therefore $E_2(\tau) - 2E_2(2\tau)$ is a modular form of weight two under elements of $\Gamma(2)$.

Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the standard generators for $PSL_2(\mathbb{Z})$. The subgroup $\Gamma(2)$ is generated by T^2 and ST^2S, see [Sch74]. The theta functions raised to the fourth power transform under S and T as:

$$\begin{aligned} \Theta_1^4(S\tau) &= \tau^2\Theta_2(\tau) \qquad & \Theta_1^4(T\tau) &= -\Theta_1^4(\tau), \\ \Theta_2^4(S\tau) &= \tau^2\Theta_1(\tau) \qquad & \Theta_2^4(T\tau) &= \Theta_3^4(\tau), \\ \Theta_3^4(S\tau) &= \tau^2\Theta_3(\tau) \qquad & \Theta_3^4(T\tau) &= \Theta_2^4(\tau), \end{aligned}$$

see [Cha85]. Using these, we can check that $\Theta_1^4(\tau)$, $\Theta_2^4(\tau)$ and $\Theta_2^4(\tau)$ are modular of weight two under $\Gamma(2)$. We know that the space of weight-two $\Gamma(2)$-forms is two dimensional, see [Gun62], and computing the q expansions of Θ_2^4 and Θ_3^4,

$$\begin{aligned} \Theta_2^4(\tau) &= 1 - 8q^{1/2} + 24q + \cdots , \\ \Theta_3^4(\tau) &= 1 + 8q^{1/2} + 24q + \cdots , \end{aligned}$$

we see that $\Theta_2^4(\tau)$ and $\Theta_3^4(\tau)$ are linearly independent. Thus $\{\Theta_2^4(\tau), \Theta_3^4(\tau)\}$ is a basis for the space of weight-two $\Gamma(2)$-forms. So $2\big(E_2(\tau) - 2E_2(2\tau)\big)$ is a linear combination of $\Theta_2^4(\tau)$ and $\Theta_3^4(\tau)$. From the definition of $E_2(\tau)$,

$$\begin{aligned} E_2(\tau) &= 1 - 24q + \cdots , \\ E_2(2\tau) &= 1 - 24q^2 + \cdots , \\ 2\big(E_2(\tau) - 2E_2(2\tau)\big) &= -2 - 48q + \cdots . \end{aligned}$$

Equation 80 follows, by equating the coefficients of q^0 and $q^{1/2}$.

To see equation (81), we apply $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to equation (80). Notice that

$$2S\tau = -\frac{2}{\tau} = -\frac{1}{\tau/2} = S(\tau/2).$$

So, from equation (83), applying S to the left hand side of equation (80) gives:

$$\begin{aligned} S \circ 2\big(E_2(\tau) - 2E_2(2\tau)\big) &= 2E_2(S\tau) - 4E_2(S(\tau/2)), \\ &= 2\Big(E_2(\tau)\tau^2 + \frac{12}{2\pi i}\tau\Big) - 4\Big(E_2(\tau/2)\frac{\tau^2}{4} + \frac{12}{2\pi i}\frac{\tau}{2}\Big), \\ &= \tau^2\big(2E_2(\tau) - E_2(\tau/2)\big). \end{aligned}$$

Applying S to the right hand side of equation 80 gives:

$$S \circ \big(-\Theta_2^4(\tau) - \Theta_3^4(\tau)\big) = \tau^2\big(\Theta_1^4(\tau) + \Theta_3^4(\tau)\big),$$

see [Cha85, ch. V, §8]. Equation (81) follows. To prove equation (82), we simply add equations (80) and (81). □

Using this proposition, the graded trace of $v(\lambda)$ in statement of Theorem 13.1 becomes:

$$\begin{aligned} Z_{V^\natural}(v(\lambda), \tau) = \frac{(\lambda,\lambda)}{8}\eta(\tau)^{12} \Bigg\{ &\big(-\Theta_2^4(\tau) - \Theta_3^4(\tau)\big)\left(\frac{\Theta_1(\tau)}{2}\right)^{(\lambda,\lambda)-12} \\ &+ \big(\Theta_1^4(\tau) + \Theta_3^4(\tau)\big)\left(\frac{\Theta_2(\tau)}{2}\right)^{(\lambda,\lambda)-12} + \big(\Theta_1^4(\tau) - \Theta_2^4(\tau)\big)\left(\frac{\Theta_3(\tau)}{2}\right)^{(\lambda,\lambda)-12} \Bigg\}. \end{aligned}$$

As we noted previously the graded trace above is zero for $(\lambda,\lambda) = 16$ or 24. Restating equation (79), using Proposition 13.9 gives:

$$\big(-\Theta_2^4(\tau) - \Theta_3^4(\tau)\big)\Theta_1(\tau)^n + \big(\Theta_1^4(\tau) + \Theta_3^4(\tau)\big)\Theta_2(\tau)^n + \big(\Theta_1^4(\tau) - \Theta_2^4(\tau)\big)\Theta_3(\tau)^n = 0,$$

where $n = 4$ or 12. Stated this way, the equality for $n = 4$ becomes obvious.

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