1. (10 points) Find a subset $U \subseteq \mathbb{R}^2$ which is closed under addition and additive inverses, but is not a subspace of $\mathbb{R}^2$.

Consider $U := \mathbb{Z} \times 0 = \{(n, 0) \mid n \in \mathbb{Z}\} \subseteq \mathbb{R}^2$ (here $\mathbb{Z}$ is the set of integers). To see that $U$ is closed under addition and additive inverses, pick $u_1, u_2 \in U$. Then $u_1 = (n_1, 0)$ and $u_2 = (n_2, 0)$ for some $n_1, n_2 \in \mathbb{Z}$, and

$$u_1 + u_2 = (n_1, 0) + (n_2, 0) = (n_1 + n_2, 0) \in U \text{ (since } \mathbb{Z} \text{ is closed under addition)},$$

$$-u_1 = -(n_1, 0) = (-n_1, 0) \in U \text{ (since } \mathbb{Z} \text{ is closed under additive inverses)}.$$

However, $(1, 0) \in U$ but $\frac{1}{2}(1, 0) = (\frac{1}{2}, 0) \notin U$, so $U$ is not closed under scalar multiplication.

2. (10 points) Let $V$ be a vector space, and consider the set of subspaces of $V$, $\text{Sub}(V) := \{U \subseteq V \mid U \text{ is a subspace of } V\}$. Define addition and scalar multiplication on $\text{Sub}(V)$ as usual, i.e. $U_1 + U_2 := \{u_1 + u_2 \mid u_1 \in U_1, u_2 \in U_2\}$ and $\lambda U := \{\lambda u \mid u \in U\}$. With these operations, is $\text{Sub}(V)$ a vector space?

No, if $V$ is not the zero vector space: note that the additive identity (= zero vector) must be the zero subspace. But then $V$ itself has no additive inverse.
3. (10 points) Let \( V \) be a vector space, and suppose \( \{ v_1, v_2, v_3, v_4 \} \) spans \( V \). Show that \( \{ v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4 \} \) also spans \( V \).

Since

\[
\begin{align*}
v_1 &= (v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + v_4, \\
v_2 &= (v_2 - v_3) + (v_3 - v_4) + v_4, \\
v_3 &= (v_3 - v_4) + v_4.
\end{align*}
\]

it follows that \( v_1, v_2, v_3, v_4 \in \text{span}\{ v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4 \} \) and hence \( V = \text{span}\{ v_1, v_2, v_3, v_4 \} \subseteq \text{span}\{ v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4 \} \).

4. Let \( U := \{ p \in P_4(\mathbb{R}) \mid \int_{-1}^{1} p \, dx = 0 \} \) (here \( P_4 \) denotes the vector space of polynomials of degree at most 4).

i) (5 points) Find (with justification) a basis of \( U \).

Note that \( U \) is a proper subspace of \( P_4(\mathbb{R}) \), since e.g. \( 1 \not\in U \), so \( \dim U \leq 4 \). Now any odd function \( f \) satisfies \( \int_{-1}^{1} f \, dx = 0 \), so in particular \( x, x^3 \) are both in \( U \). By inspection, we also see that \( 3x^2 - 1, 5x^4 - 1 \) (which integrate to \( x^3 - x, x^5 - x \) respectively) are in \( U \). Moreover, \( \{ x, 3x^2 - 1, x^3, 5x^4 - 1 \} \) is linearly independent: if \( a_1 x + a_2 (3x^2 - 1) + a_3 x^3 + a_4 (5x^4 - 1) = 0 \) for some \( a_i \in \mathbb{R} \), then by comparing coefficients of \( x^4 \) (resp. \( x^3, x^2, x \)) one has \( 5a_4 \) (resp. \( a_3, 3a_2, a_1 \)) equal to 0, and thus all \( a_i \) must be zero. This shows \( \dim U \geq 4 \), so in fact equality holds and \( \{ x, 3x^2 - 1, x^3, 5x^4 - 1 \} \) is a basis of \( U \).

ii) (5 points) Extend the basis in (a) to a basis of \( P_4(\mathbb{R}) \).

Since \( \dim P_4(\mathbb{R}) = 5 \), it suffices to find a polynomial \( p \in P_4(\mathbb{R}) \) such that \( \{ x, 3x^2 - 1, x^3, 5x^4 - 1, p \} \) is linearly independent (as any linearly independent set of cardinality equal to \( \dim P_4(\mathbb{R}) \) is a basis of \( P_4(\mathbb{R}) \)). One sees that \( p := 1 \) works: if \( a_0 (1) + a_1 x + a_2 (3x^2 - 1) + a_3 x^3 + a_4 (5x^4 - 1) = 0 \) for some \( a_i \in \mathbb{R} \), then the same reasoning as in (i) implies \( a_4 = a_3 = a_2 = a_1 = 0 \), and substituting yields \( a_0 = 0 \) as well.
5. Let $T_1: V \to W$, $T_2: U \to V$, be linear maps, and suppose that $T_2 \circ T_1$ is surjective. Prove or give a counterexample to the following statements:

i) (5 points) $T_1$ is surjective

Take $U = \mathbb{R}, V = \mathbb{R}^2, W = \{0\}$. Then any linear map into $W$ is surjective, but there is no surjection from $U$ onto $V$.

ii) (5 points) $T_2$ is surjective.

Since $\text{im}(T_2 \circ T_1) = \{T_2(T_1(u)) \mid u \in U\} \subseteq \{T_2(v) \mid v \in V\} = \text{im}(T_2)$, one has $T_2 \circ T_1$ surjective $\implies$ $\text{im}(T_2) = W$ $\implies$ $\text{im}(T_2) = W$ $\implies$ $T_2$ surjective.

6. (10 points) Prove or give a counterexample: if $V$ is a finite-dimensional vector space and $U_1, U_2, U_3$ are subspaces of $V$, then

$$\dim(U_1 + U_2 + U_3) = \dim(U_1) + \dim(U_2) + \dim(U_3)$$
$$- \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3)$$
$$+ \dim(U_1 \cap U_2 \cap U_3)$$

Consider $V := \mathbb{R}^2$, $U_1 := \text{span}\{(1,0)\}$, $U_2 := \text{span}\{(0,1)\}$, $U_3 := \text{span}\{(1,1)\}$. Then $U_i \cap U_j = \{0\}$ for $i \neq j$ (which implies $U_1 \cap U_2 \cap U_3 = \{0\}$ and $U_1 + U_2 + U_3 = \mathbb{R}^2$), so the statement becomes $2 = 1 + 1 - 0 - 0 + 0$, which is false.
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