1. (10 points) Let $V$ be a vector space of dimension 10, and let $T \in \mathcal{L}(V)$ with $\dim \ker T \geq 6$. Show that $T$ has at most 5 distinct eigenvalues.

2. (10 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by

$$T(x_1, x_2, x_3) = (2x_1 + x_2 + x_3, x_1 + 2x_2 + x_3, x_1 + x_2 + 2x_3)$$

Find all eigenvalues of $T$. Is $T$ diagonalizable? (Hint: first show that $T$ is diagonalizable iff $T + I$ is diagonalizable, then consider the matrix whose entries are all 1.)
3. (10 points) Let $V$ be an inner product space, and let $u, v \in V$. Show that $||u|| = ||v||$ iff $||au + bv|| = ||bu + av||$ for all $a, b \in \mathbb{R}$.

4. (10 points) Let $\mathcal{P}_2(\mathbb{R})$ denote the vector space of real polynomials of degree at most 2, with inner product $\langle p, q \rangle := \int_0^1 p(x)q(x) \, dx$. Apply the Gram-Schmidt procedure to obtain an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$ from the basis

$$\{x^2, x, 1\}$$
5. (10 points) Let \( a, b, c, d \) be positive real numbers. Show that \((a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \geq 16.\)

6. (10 points) Let \( V \) be a vector space, \( T \in \mathcal{L}(V) \), and suppose \( u, v \in V \) are eigenvectors of \( T \) such that \( u + v \) is also an eigenvector of \( T \). Show that \( u, v \) are eigenvectors of \( T \) corresponding to the same eigenvalue.
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