GEORGIA TECH

MATH, PHYSICS & COMPUTING
MATH 4803BDU,BDG, CS4803F, PHYS4803A

QUANTUM INFORMATION & QUANTUM COMPUTING

Final Exam

April 30th, 2004

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Quantum circuits I : Give the expression of the unitary matrices representing the 1-qubit gates denotes by X, Y, Z, H, S, T.

$$X =$$

$$Y =$$

$$Z =$$

$$S =$$

$$T =$$

Quantum circuits II : Give the design and the action of the CNOT and the Toffoli gates for input given in the digital basis. Give the corresponding unitary matrices U_{CNOT} and U_{Toff}.

CNOT gate

$$U_{CNOT} =$$

Toffoli gate

$$U_{Toff} =$$

Quantum circuit III Compute the outpout of the circuit given in Fig. 1 below.

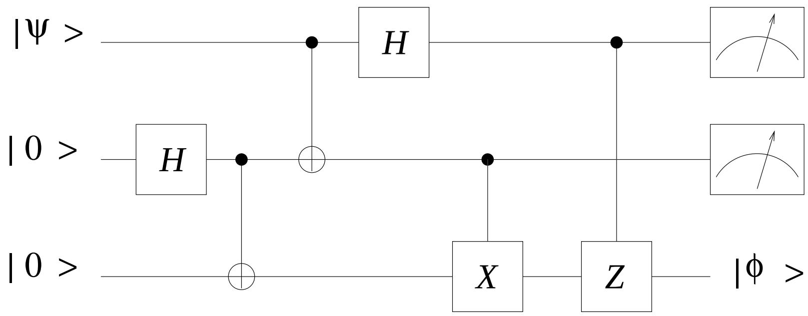

FIG. 1 – What is the output $|\phi\rangle$?.

Quantum circuit IV Explain and justify what is the algorithm produced by the circuit given in Fig. 2 below.

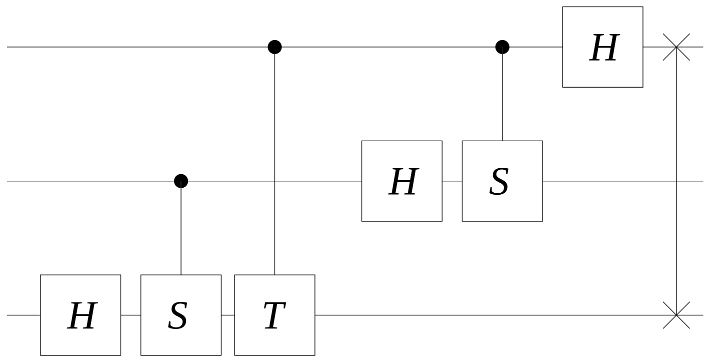

FIG. 2 – A 3-qubits circuit

Quantum Mechanics I : (i) Give the expression of the matrix $Z \otimes X \otimes X$

$$X \otimes Y \otimes Z =$$

(ii) Give the expression of the 2-qubit Bell state $|\beta_{00}\rangle$.

$$|\beta_{00}\rangle =$$

(iii) Compute the partial trace (over the 2nd qubit) of the pure state $|\beta_{00}\rangle\langle\beta_{00}|$

$$\mathrm{Tr}_2\left(|\beta_{00}\rangle\langle\beta_{00}|\right) =$$

Quantum Mechanics II : In a 2-qubits system, let X_i, Y_i, Z_i be the Pauli matrices acting on the qubit $i \in \{1, 2\}$. Then H denotes the operator $H = -J(X_1X_2 + Y_1Y_2 + Z_1Z_2) - B(Z_1 + Z_2)$ with $J, B > 0$.

1. Compute the states $H|x_1, x_2\rangle$, if $|x_1, x_2\rangle$ denotes the computational basis.

2. Compute the eigenvalues of H.

3. Give the smallest eigenvalue (the so-called *groundstate energy*) as a function of B

4. Give an orthonormal basis of eigenvectors of H.

Quantum Measurement I :

1. What are the three mathematical properties characterizing a *density matrix*?

2. Which one of the following ρ_i's is a density matrix? (*Explain why*)

$$\rho_1 = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad \rho_2 = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ +1 & 1 \end{bmatrix} \quad \rho_3 = \frac{1}{3}\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \quad \rho_4 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Quantum Measurement II :

1. What is the mathematical definition of a *measurement process* $\underline{M}$? In particular
   (a) what is an outcome ?
   (b) what is a measurement operator ?
   (c) what property the measurement operator must satisfy ?

2. Let $\underline{M}$ be a measurement process :
   (a) what is the probability of an outcome if the initial state before measurement is given by the density matrix ρ ?
   (b) what is the density matrix after measurement if an outcome has been obtained ?
   (c) what is the density matrix after measurement in absence of outcome ?

3. What is the mathematical definition of a quantum operation ? (*Hint : give the definition of complete positivity*)

4. Given a measurement process $\underline{M}$, what is the quantum operation associated with it ?

5. Let $H = H^{\dagger}$ be a selfadjoint operator and let $f : \mathbb{R} \mapsto \mathbb{C}$ be a function. What are the conditions on the function f to make $\mathfrak{L}$ below a quantum operation ?

$$\mathfrak{L}(\rho) \;=\; \int_{-\infty}^{+\infty} dt f(t) \;\; e^{\imath tH} \rho e^{-\imath tH}$$

Quantum measurement III :

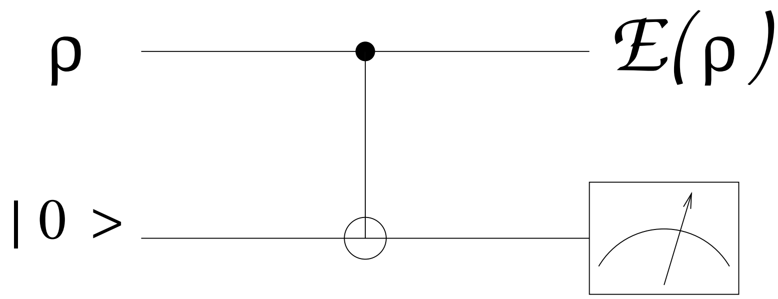

FIG. 3 – The CNOT gate as a quantum operation.

1. Compute $U_{CNOT}|x0\rangle$.

2. If ρ is a 2×2 density matrix shown below describing the first qubit state, what is the density matrix $\hat{\rho}$ of the input shown in Fig. 3 above? (*Hint : use the matrices* $|xy\rangle\langle x'y'|$ *to express the result.*)

$$\rho \;=\; \frac{1}{3}\begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix}$$

3. Compute $U_{CNOT}\,\hat{\rho}\,U_{CNOT}^{\dagger}$. (*Hint : use the matrices* $|xy\rangle\langle x'y'|$ *to express the result.*)

4. Compute the CNOT quantum operation $\mathcal{E}$ given by Fig. 3 above (*Hint : use partial trace and express the result using* $P_x = |x\rangle\langle x|$ *where* $x \in \{0, 1\}$.)

5. Compute $\mathcal{E}_1$ shown below and compare with $\mathcal{E}$

$$\mathcal{E}_1(\rho) \;=\; \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{\imath\theta Z}\; \rho\, e^{-\imath\theta Z}$$

The Shor code :

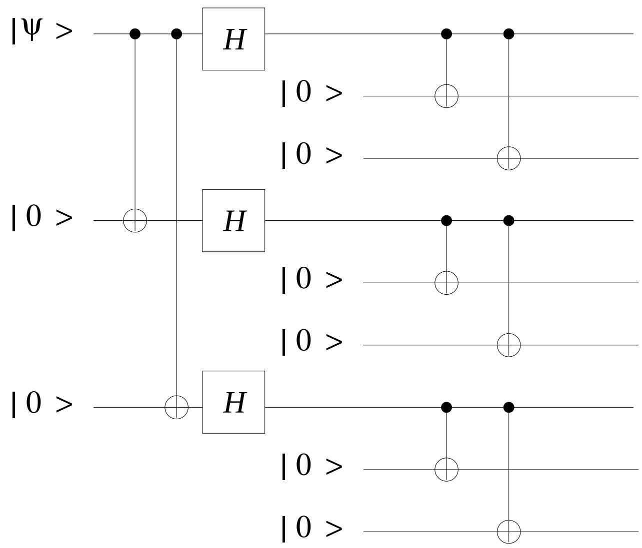

FIG. 4 – Encoding circuit for the Shor 9-qubit code.

Coding one qubit with nine qubit through the Shor code can be represented by the circuit designed in Fig. 4 above. If $i = 1, \cdots, 9$ index the qubit from top to bottom, let X_i, Z_i denote the Pauli matrices acting on the i-th qubit.

1. Compute the output whenever $|\psi\rangle = |0\rangle$ or $|1\rangle$.(*the corresponding output will be denotes by* $|0_L\rangle$ *and* $|1_L\rangle$ *respectively.*)

2. Show that the syndrome measurement for detecting *bit flip* errors on either of the top three qubits corresponds to measuring the observables Z_1Z_2 and Z_2Z_3. Indicate how to detect bit flips of other qubits.

3. Let $m \in \{0, 1, 2, 3\}$ be the outcome of the previous measurement, where $m = 0$ if no qubit has been flipped and $m = i \in \{1, 2, 3\}$ if the qubit i (or if the two qubits in $\{1, 2, 3\} \setminus \{i\}$) has been flipped. Give the corresponding eigenvalues of the pair $\{Z_1Z_2, Z_2Z_3\}$ for each values of m.

4. Describe how to recover from a bit flip by using m and the observables X_1, X_2, X_3.

5. Show that the syndrome measurement for detecting *phase flip* errors corresponds to measuring the observables $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$.

6. Show that the operators $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$ commute with the Z_i. Conclude that there is aprojective measurement process that detects both bit flips and phase flips.

7. Show that recovery from a phase flip on any of the first three qubits, may be accomplished by applying the operator $Z_1Z_2Z_3$.