Georgia Tech

Math, Physics & Computing
Math 4782BDU,BDG, Phys 4782A, CS4803QIC

# Quantum Information & Quantum Computing

## Final Exam

*May 1st, 2007, 11 :00am-2 :40pm, Skiles 246*

**First Name :** ______________________________________________

**Name :** __________________________________________________

**Gatech Id # :** _____________________________________________

**e-mail :** __________________________________________________

**Quantum circuits I :** Give the expression of the unitary matrices representing the 1-qubit gates denotes by $X, Y, Z, H, S, T$.

$$X =$$

$$Y =$$

$$Z =$$

$$S =$$

$$T =$$

**Quantum circuits II :** Give the design and the action of the Toffoli gates for input given in the digital basis. Give the corresponding unitary matrice $U_{Toff}$

Toffoli gate

$$U_{Toff} =$$

**Quantum circuit III** Compute the outpout of the circuit given in Fig. 1 below. (*Details of this computation will be appreciated.*)

FIG. 1 – What is the output $|\phi\rangle$ ?.

**Quantum circuit IV** Show that the circuit given in Fig. 1 below gives the Fourier transform for 3-qubits.

FIG. 2 – Fourier circuit for 3-qubits ?

**Quantum Mechanics I :** (i) Give the expression of the matrix $X \otimes Y \otimes Z$

$$X \otimes Y \otimes Z =$$

(ii) Give the expression of the 2-qubit Bell state $|\beta_{00}\rangle$.

$$|\beta_{00}\rangle =$$

(iii) Compute the partial trace (over the 2nd qubit) of the pure state $|\beta_{00}\rangle\langle\beta_{00}|$

$$\mathrm{Tr}_2\left(|\beta_{00}\rangle\langle\beta_{00}|\right) =$$

**Quantum operation :**

FIG. 3 – The CNOT gate as a quantum operation.

1. Show that $U_{CNOT} = P_0 \otimes I + P_1 \otimes X$, if $P_x = |x\rangle\langle x|$.

2. Compute the CNOT quantum operation $\mathcal{E}$ given by Fig. 3 above.

**The Steane code :**

Steane proposed a quantum codes encoding one qubit out of seven qubits, instead of nine for the Schor code. It uses two linear classical codes $C_1, C_2$ with $C_2 = C_1^{\perp} \subset C_1$ corresponding to the generator matrices $G_1, G_2$ and parity check matrices $H_1, H_2$ as follows

$$0 \longrightarrow \mathbb{Z}_2^4 \xrightarrow{G_1} \mathbb{Z}_2^7 \xrightarrow{H_1} \mathbb{Z}_2^3 \longrightarrow 0$$

$$0 \longleftarrow \mathbb{Z}_2^4 \xleftarrow{H_2} \mathbb{Z}_2^7 \xleftarrow{G_2} \mathbb{Z}_2^3 \longleftarrow 0$$

where the code $C_2$ is obtained from $C_1$ by transposing all arrows giving $H_2 = G_1^t$ , $G_2 = H_1^t$, so that $C_2 = \mathrm{Ker} H_2 = \mathrm{Ker} G_1^t = \mathrm{Im} {G_1}^{\perp} = C_1^{\perp}$. As a reminder, $C_i = \mathrm{Im} G_i = Ker H_i$ for $i = 1, 2$. The Steane code is constructed with a matrix $H_1$ of the form $[\mathbf{1}_3|A]$ where $A$ is a $3 \times 4$ matrix with entries in $\mathbb{Z}_2$.

1. **The classical codes $C_1, C_2$ :**
   (a) Show that $G_1$ has the form $G_1 = \begin{bmatrix} A \\ \mathbf{1}_4 \end{bmatrix}$. (*Hint : compute* $\mathrm{Ker} H_1$*, namely the solutions in* $\mathbb{Z}_2^7$ *of the equation* $H_1 x = 0$. )
   (b) Compute then $H_2, G_2$ in terms of the matrix $A$.
   (c) Show that $C_2 \subset C_1$ if and only if $AA^t = \mathbf{1}_3$
   (d) The matrix $H_1$ is built so that its column are made of the binary digits of the numbers $1, \cdots, 2^3 - 1$. Show that if $H_1$ is presented in the form $[\mathbf{1}_3|A]$, then A can be chosen as

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

   (e) Show that $AA^t = \mathbf{1}_3$
   (f) What are the dimensions of $C_1$ and $C_2$. How many points have these two $ZM_2$-vector spaces ? Show that $C_1/C_2$ is a vector space over $\mathbb{Z}_2$ of dimension 1. How many points are in such a space ?
   (g) Show that $\Pi = A^t A$ is a *projector*, namely $\Pi^2 = \Pi$. (*Hint : use* $AA^t = \mathbf{1}_3$.)
   (h) Show that if $y \in \mathbb{Z}_2^4$ satisfies $\Pi y = 0$ then $x_0 = G_1 y \in C_1$ does not belong to $C_2$. Conclude that $C_1/C_2 = \{a[x_0]\,;\, a = 0, 1\}$ where $x_0 + C_2 = [x_0]$ is the class of $x_0$ in $C_1/C_2$. Compute explicitely the components of $x_0$.

2. **The quantum Steane code :**
   (a) The *quantum* code space defined by the Steane code is the complex Hilbert subspace of the 7-qubits Hilbert space defined by the basis $\{|0\rangle_S, |1\rangle_S\}$ with

$$|a\rangle_S = \frac{1}{\sqrt{|C_2|}} \sum_{y \in C_2} |ax_0 + y\rangle \qquad\qquad a = 0, 1\,.$$

   Show that $y$ can be written as $y_1 w_1 + y_2 w_2 + y_3 w_3$ where $y_i = 0, 1$ and the $w_i$'s are the columns of matrix $G_1$ (namely the code words for $C_1$).

(b) Using the same idea than is the Shor code, design a quantum circuit starting with $|a\rangle$ and producing $|ax_0\rangle$ in the 7-qubits space.

(c) What quantum gate produces $(|0\rangle + |1\rangle)/\sqrt{2}$ ?

(d) Give a simple quantum circuit producing $|ax_0 + w_1\rangle$ ? Same question with $w_2, w_3$.

(e) Conclude that the circuit of Fig. 4 below produces the Steane code out of the input $|a\rangle$ , $a = 0, 1$.

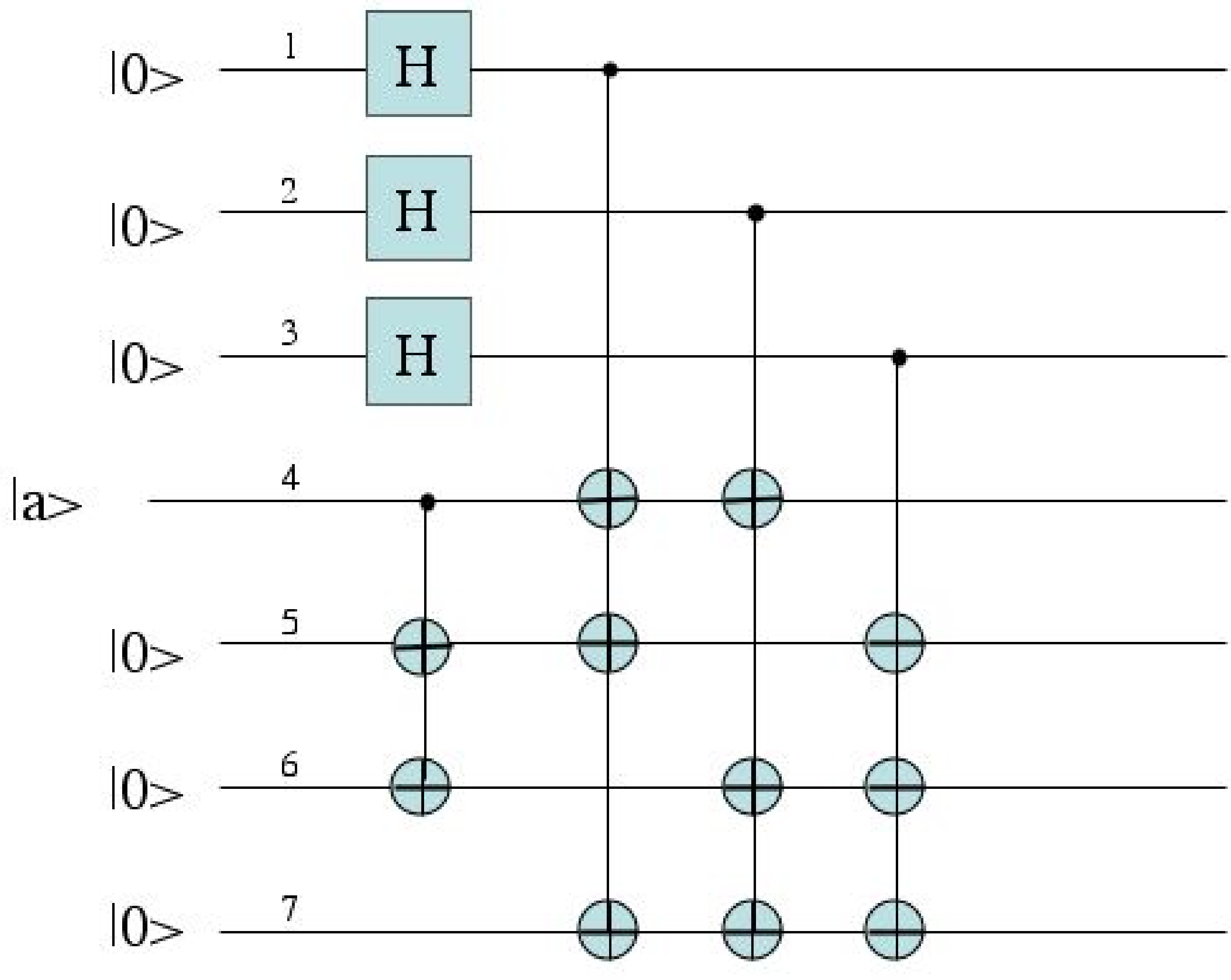


FIG. 4 – A quantum circuit for the Steane code