Elementary Quantum Gates:

1. The gate CNOT is represented in Fig. 1(left). Give the expression of the unitary operator associated with it.

2. If, in the CNOT gate above, \( b = 0 \) and the state \(|a\rangle\) is replaced by an unknown state \(|\psi\rangle = \alpha|0\rangle + \beta|1\rangle\), compute the resulting state. Can such a gate be used to copy the state \(|\psi\rangle\)? Explain why.

3. The SWAP gate is the association of three CNOT gates as in Fig. 1(center left). Given the incoming state \(|a\rangle \langle b|\)

4. What is the Hadamard matrix \( H \) and the corresponding gate? Show that the circuit represented in Fig. 2(center right) creates Bell states: give the expression of \(|\beta_{xy}\rangle\) for all possible values of the bits \(x, y\).

5. In the 2-qbits space, let \(S_1, S_2\) be the operators defined by \(S_1|xy\rangle = x|xy\rangle\) and \(S_2|xy\rangle = y|xy\rangle\). Show that

\[
\langle\psi|(S_1 - S_2)^2\psi\rangle = 0, \quad \text{for} \quad \psi = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.
\]

Conclude that the two bits \(x\) and \(y\) are correlated if measured in the state \(\psi\) above.

6. Give the family of outputs of the Toffoli gate represented in Fig. 2(right) for all possible values of the input bits \(x, y, z\). Give the matrix of the corresponding unitary operator.
7. Using the Toffoli gate represented in Fig. 2(right), compute the output whenever the input bit $z$ is set to 0 or 1.

Quantum Teleportation:

The circuit represented in Fig. 3 is the union of two systems: the left part belongs to Alice’s system, the right one to Bob’s. The input is provided by an unknown state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ that Alice wants to send to Bob, and by the Bell state $\beta_{00} = (1/\sqrt{2})(|00\rangle + |11\rangle)$. After a CNOT and a Hadamard gate, Alice measures the signal and gets two bits denoted by $M_1, M_2$. She sends the result of this measure to Bob. Bob then proceeds to transform the state he gets on his line according to the two gates $X^{M_2}$ first, then $Z^{M_1}$, where $X$ and $Z$ are the Pauli matrices.

1. Compute the states $|\psi_j\rangle$ for $j = 0, 1, 2, 3$ obtained successively as indicated in Fig. 3.

2. Show then that the last two gates permit Bob to recover the state $|\psi\rangle$. (*Warning: be careful to apply the last two gates in the correct order!*)

3. In such an experiment, what prevents the teleportation to transfer the information contained in the input at a speed larger than the speed of light?

4. Apply the same analysis to the circuit in Fig. 4 and compute the state $|\phi\rangle$.

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