GEORGIA TECH

MATH, PHYSICS & COMPUTING
MATH 4803BDU,BDG, CS4803F, PHYS4803A

QUANTUM INFORMATION & QUANTUM COMPUTING

Problems Set X
Due

Continuous fraction expansion :

If $0 < \alpha < 1$ then let $a_1 = a(\alpha)$ denote the integer part of $1/\alpha$ and let $\alpha_1 = G(\alpha)$ denote the fractional part of $1/\alpha$. $G : (0, 1] \to [0, 1]$ is called the *Gauss map*. Then G^n will denote the n^{th} iterated of G, defined recursively by $G^n(\alpha) = G^{n-1}(G(\alpha))$. Moreover, a_n will denote the integer part of $1/G^{n-1}(\alpha)$ namely, $a_n = a(G^{n-1}(\alpha))$.

1. Draw the graph of the Gauss map.
2. Show that if $\alpha_n = G^n(\alpha)$, then

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_n + \alpha_n}}} . \tag{1}$$

   Remark : *The previous formula (1) is called the* continuous fraction expansion *of* α *and it is denotes by* $\alpha = [a_1, a_2, \cdots, a_n, \cdots]$. *Setting* $\alpha_n = 0$ *gives a fraction denoted by* p_n/q_n *and called a* convergent *of* α.

3. Use a handpocket calculator to solve this question
   (a) Compute the continuous fraction expansion of $\alpha = 271/391$ (*Remark : this is a convergent of* $\ln 2$).
   (b) Make the algorithm explicit.
   (c) If α is rational, does an ordinary handpocket calculator give the exact result ? If not explain why ?
4. The *golden mean* is the number $0 < \sigma = (\sqrt{5} - 1)/2 < 1$.
   (a) Show that $1 + \sigma = 1/\sigma$.
   (b) Deduce its continuous fraction expansion.
   (c) It will be shown later (see question (7e)) that its convergent are given by F_{n-1}/F_n where $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$. The F_n's are called the *Fibonacci numbers*. Give the explicit values of the F_n's for $n \le 11$.
   (d) Prove, by recursion, that $F_n \ge (1/\sigma)^{n-1}$.
   (e) The result proved in question (7e) show that if p_n/q_n is the n-th convergent of α, then $q_n \ge F_n$. Conclude that the algorithm giving the continuous fraction expansion of p/q terminates after $n = O(\ln q)$ steps.
   Remark : *It can be proved that each such step requires* $O((\ln q)^2)$ *gates of elementary arithmetic.*

5. Show that $\alpha = p/q$ is a fraction if and only if there is an n such that $\alpha_n = 0$ (*Hint : show that if α is a fraction, so is $G(\alpha)$*).
6. A map $F : \mathbb{R} \mapsto \mathbb{R}$ is called *harmonic*, whenever it is of the form

$$F(x) = F_A(x) = \frac{p+rx}{q+sx}, \qquad rq - sp = \pm 1 \qquad A = \begin{bmatrix} q & s \\ p & r \end{bmatrix},$$

for some real numbers p, q, r, s.

   (a) Show that the composition of F_A and F_B is the map F_{AB}. In particular the composition of two harmonic maps is harmonic.
   (b) If A and B have integral entries, show that so does AB.
   (c) What values are taken on by $\det A$?

7. If a is an integer larger than or equal to 1, let f_a be the map $f_a : x \in [0,1] \mapsto 1/(a+x) \in [0,1]$.
   (a) Show that f_a is harmonic and give the expression of the corresponding matrix that will be denoted by T_a.
   (b) Show that the continuous fraction decomposition of α can be expressed as

$$\alpha = f_{a_1} \circ f_{a_2} \circ \cdots \circ f_{a_n}(\alpha_n) \tag{2}$$

   (c) Conclude that there are integers p_n, q_n, r_n, s_n such that

$$\alpha = \frac{p_n + r_n \alpha_n}{q_n + s_n \alpha_n}, \tag{3}$$

   and gives the corresponding matrix A_n in term of a product of the T_{a_k}'s.
   (d) Compute p_0, q_0, r_0, s_0.
   (e) Show recursively that

$$r_n = p_{n-1}\,,\ s_n = q_{n-1}\,, \qquad p_{n+1} = a_{n+1}p_n + p_{n-1}\,, \qquad q_{n+1} = a_{n+1}q_n + q_{n-1}\,.$$

   (f) Show that $q_n p_{n-1} - q_{n-1} p_n = (-1)^n$. In particular show that p_n and q_n are prime to each other.
   (g) Prove from eq. (2) that α is an increasing function of α_n if n is even, and is decreasing if n is odd.
   (h) Conclude that

$$\frac{p_{2m}}{q_{2m}} \leq \frac{p_{2m+2}}{q_{2m+2}} \leq \alpha \leq \frac{p_{2m+1}}{q_{2m+1}} \leq \frac{p_{2m-1}}{q_{2m-1}}.$$

   (i) Show that

$$|\alpha - \frac{p_n}{q_n}| \leq |\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}| = \frac{1}{q_n q_{n+1}} \leq \frac{1}{q_n^2}.$$

8. Let now p and q be coprime integers such that $p < q$. Using the continuous fraction expansion of p/q, prove the *Bezout* theorem, namely, that there are $p', q' \in \mathbb{Z}$ such that $pq' + qp' = 1$ integers.
9. Let now p and q be coprime integers such that $|\alpha - p/q| < 1/2q^2$. In this question it will be proved that p/q is a convergent of α.
   (a) Show that $|\alpha - p/q| < 1/(2q^2 - q)$ and that $|\alpha - p/q| < 1/2$.
   (b) If $p/q = 1$, show that there is $\alpha' \in (0, 1)$ such that
   $$\alpha = \frac{1}{1 + \alpha'}.$$
   Conclude then that 1 is a convergent for α.
   (c) Assume now that $p = 1$ and $q \geq 2$. Using the question (9a), show that one of the two possibilities occurs :
   $$(i) \quad q \leq \frac{1}{\alpha} < \frac{q}{1 - \frac{1}{2q-1}}, \qquad (ii) \quad \frac{q}{1 + \frac{1}{2q-1}} < \frac{1}{\alpha} < q.$$
   Then show that there is $0 \leq \alpha' < 1$ such that
   $$\text{in case } (i) \quad \alpha = \frac{1}{q + \alpha'}, \qquad \text{in case } (ii) \quad \alpha = \frac{1}{q - 1 + \frac{1}{1+\alpha'}}.$$
   Then conclude that in both cases, $1/q$ is a convergent of α.
   (d) Assume now that $2 \leq p < q$. Then let $a \geq 1$ and $0 \leq r \leq p - 1$ be the ratio and the rest in the Euclidean division of q by p, namely $q = ap + r$. Since p and q are coprime, conclude that $1 \leq r \leq p - 1$. Using the question (9a), show that there is $\alpha' \in (0, 1)$ such that
   $$\alpha = \frac{1}{a + \alpha'}, \qquad \text{with} \qquad |\alpha' - \frac{r}{p}| < \frac{1}{2p^2 - p}.$$
   (e) Using the previous argument recursively, conclude that p/q is a convergent of α.