The GAP LABELLING THEOREM for TILINGS

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Main References


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I - The Hull as a Dynamical System

J. Bellissard, D. Hermann, M. Zariouati, Hull of Aperiodic Solids and Gap Labelling Theorems
I.1)- Point Sets

Equilibrium positions of atomic nuclei make up a point set $\mathcal{L} \subset \mathbb{R}^d$ the set of lattice sites. $\mathcal{L}$ may be:

1. Discrete.

2. Uniformly discrete: $\exists r > 0$ s.t. each ball of radius $r$ contains at most one point of $\mathcal{L}$.

3. A Delone set: $\mathcal{L}$ is uniformly discrete and relatively dense: $\exists R > 0$ s.t. each ball of radius $R$ contains at least two points of $\mathcal{L}$.

4. A Meyer set: $\mathcal{L}$ and $\mathcal{L} - \mathcal{L}$ are Delone sets.

Examples:

1. A random Poissonian set in $\mathbb{R}^d$ is almost surely discrete but not uniformly discrete nor relatively dense.

2. Due to Coulomb repulsion and Quantum Mechanics, lattices of atoms are always uniformly discrete.

3. Impurities in semiconductors are not relatively dense.

4. In amorphous media $\mathcal{L}$ is Delone.

5. In a quasicrystal $\mathcal{L}$ is Meyer.
I.2) Point Measures

\[ \mathcal{M}(\mathbb{R}^d) \] is the set of Radon measures on \( \mathbb{R}^d \) namely the dual space to \( \mathcal{C}_c(\mathbb{R}^d) \) (continuous functions with compact support), endowed with the weak* topology.

For \( \mathcal{L} \) a uniformly discrete point set in \( \mathbb{R}^d \):

\[ \nu := \nu^\mathcal{L} = \sum_{y \in \mathcal{L}} \delta(x - y) \in \mathcal{M}(\mathbb{R}^d) . \]

The Hull is the closure in \( \mathcal{M}(\mathbb{R}^d) \)

\[ \Omega = \overline{\{ T^a \nu^\mathcal{L} ; a \in \mathbb{R}^d \}} , \]

where \( T^a \nu \) is the translated of \( \nu \) by \( a \).

**Facts:**

1. \( \Omega \) is compact and \( \mathbb{R}^d \) acts by homeomorphisms.
2. If \( \omega \in \Omega \), there is a uniformly discrete point set \( \mathcal{L}_\omega \) in \( \mathbb{R}^d \) such that \( \omega \) coincides with \( \nu_\omega = \nu^\mathcal{L}_\omega \).
3. If \( \mathcal{L} \) is Delone (resp. Meyer) so are the \( \mathcal{L}_\omega \)'s.
I.3)- Properties

(a) Minimality

**Proposition 1** $\mathbb{R}^d$ acts minimally on $\Omega$ if and only if, for any $\omega \in \Omega$ and $F \subseteq \Omega$ closed, the subset $\mathcal{L}^F_\omega = \{x \in \mathcal{L}_\omega; T^{-x}\omega \in F\}$ is a Delone set.

(b) Transversal

The closed subset $X = \{\omega \in \Omega ; \nu_\omega(\{0\}) = 1\}$ is called the *canonical transversal*. Let $G$ be the sub-groupoid of $\Omega \rtimes \mathbb{R}^d$ induced by $X$.

A Delone set $\mathcal{L}$ has *finite type* if $\mathcal{L} - \mathcal{L}$ is closed and discrete.

(c) Cantorian Transversal

**Proposition 2** If $\mathcal{L}$ has finite type, then the transversal is completely discontinuous (Cantor).
II - Tilings & Point Sets


II.1)- Voronoi Cells

For $\mathcal{L}$ Delone and $x \in \mathcal{L}$, the Voronoi cell of $x$ is

$$V_x = \{ y \in \mathbb{R}^d ; |y - x| < |y - x'|, \forall x' \in \mathcal{L} \setminus \{x\} \}$$

- Building a Voronoi Cell -

The $V_x$’s are open polyhedrons with uniformly bounded diameter. They are mutually disjoint and their closure cover $\mathbb{R}^d$: it is a tiling of $\mathbb{R}^d$
II.2)- The Finite Pattern Condition

A tiling $\mathcal{T}$ fulfills the *finite pattern condition* (FPC) if the number of tiles *modulo translations* is finite. A *patch* is the set of tiles of $\mathcal{T}$ contained in some ball. The number of patches of given radius is finite iff $\mathcal{T}$ is FPC. Then the Hull is transversally Cantor.

- The octagonal tiling is FPC -
- The pinwheel tiling is NOT FPC ! -
II.3)- Branched Oriented Flat Manifolds

Step 1:

1. $X$ is the disjoint union of all prototiles;
2. glue prototiles $T_1$ and $T_2$ along a face $F_1 \subset T_1$ and $F_2 \subset T_2$ if $F_2$ is a translated of $F_1$ and if there are $x_1, x_2 \in \mathbb{R}^d$ such that $x_i + T_i$ are tiles of $\mathcal{T}$ with $(x_1 + T_1) \cap (x_2 + T_2) = x_1 + F_1 = x_2 + F_2$;
3. after identification of faces, $X$ becomes a branched oriented flat manifold (BOF) $B_0$.

- The branching process -
- Vertex branching for the octagonal tiling -
Step 2:

1. Choose an increasing sequence \( \{R_n\}_{n>0} \) of positive real numbers with \( R_n \uparrow \infty \);
2. for each \( n \geq 1 \) consider all patches of diameter less than \( R_n \);
3. add to each patch in \( \mathcal{T} \), the tiles touching it from outside along its frontier. Call such a patch \textit{modulo translation} a \textit{a colored patch};
4. proceed then as in Step 1 by replacing prototiles by colored patches to get the BOF-manifold \( B_n \).

- A colored patch -
Step 3:

1. Define a **BOF-submersion** \( f_n : B_{n+1} \to B_n \) by identifying patches of order \( n \) in \( B_{n+1} \) with the prototiles of \( B_n \);

2. call \( \Omega \) the **projective limit** of the sequence

\[
\cdots \xrightarrow{f_{n+1}} B_{n+1} \xrightarrow{f_n} B_n \xrightarrow{f_{n-1}} \cdots
\]

3. there are commuting vector fields \( X_1, \cdots X_d \) on \( B_n \) generating local translations and giving rise to a \( \mathbb{R}^d \) action \( \mathcal{T} \) on \( \Omega \).

**Theorem 1** The dynamical system

\[
(\Omega, \mathbb{R}^d, \mathcal{T}) = \lim (B_n, f_n)
\]

obtained as inverse limit of branched oriented flat manifolds, is conjugate to the Hull of the Delone set of the tiling \( \mathcal{T} \) by an homomorphism.
III - Gap Labelling and $K$-theory

III.1)- Algebra

Let $\mathcal{A}_0 = C_c(\Omega \times \mathbb{R}^d)$ be seen as a dense subalgebra of $\mathcal{A} = C(\Omega) \times \mathbb{R}^d$. For any $\omega \in \Omega$, let $\pi_\omega$ be the left regular representation on $\mathcal{H} = L^2(\mathbb{R}^d)$ defined by:

$$\pi_\omega(A)\psi(x) = \int_{\mathbb{R}^d} d^dy \ A(T^{-x}\omega, y - x) \ \psi(y),$$

and $\psi \in \mathcal{H}$.

If $\mathbb{P}$ is an $\mathbb{R}^d$-invariant ergodic probability measure on $\Omega$, let $T_\mathbb{P}$ be the trace on $\mathcal{A}$ defined by

$$T_\mathbb{P}(A) = \int_{\Omega} \mathbb{P}(d\omega) \ A(\omega, 0),$$

for $A \in \mathcal{A}_0$. 
III.2)- Hamiltonian

Schrödinger’s equation (ignoring interactions)

\[ H_\omega = -\frac{\hbar^2}{2m}\Delta + \sum_{y \in \mathcal{L}_\omega} v(. - y), \]

acting on \( \mathcal{H} = L^2(\mathbb{R}^d) \). Here \( v \in L^1(\mathbb{R}^d) \) is real valued, decays fast enough, is the atomic potential.

**Proposition 3** There is \( R(z) \in \mathcal{A} \), such that, for every \( \omega \in \Omega \) and \( z \in \mathbb{C} \setminus \mathbb{R} \)

\[ (z - H_\omega)^{-1} = \pi_\omega(R(z)). \]

If \( \Sigma_H = \bigcup_{\omega \in \Omega} \text{Sp}(H_\omega) \), then \( R(z) \) is holomorphic in \( z \in \mathbb{C} \setminus \Sigma_H \).

The bounded connected components of \( \mathbb{R} \setminus \Sigma_H \) are called spectral gaps.
III.3)- Density of States

- Let $\mathbb{P}$ be an invariant ergodic probability on $\Omega$. Let

$$\mathcal{N}(E) = \lim_{\Lambda \uparrow \mathbb{R}^d} \frac{1}{|\Lambda|} \# \{\text{eigenvalues of } H_\omega|_\Lambda \leq E\}$$

It is called the Integrated Density of states or IDS.

- The limit above exists $\mathbb{P}$-almost surely and

$$\mathcal{N}(E) = \mathcal{F}_\mathbb{P}(\chi(H \leq E)) \quad (\text{Shubin, '76})$$

$\chi(H \leq E)$ is the eigenprojector of $H$ in $\mathcal{L}^\infty(\mathcal{A})$.

- $\mathcal{N}$ is non decreasing, non negative and constant on gaps. $\mathcal{N}(E) = 0$ for $E < \inf \Sigma_H$. For $E \to \infty$, $\mathcal{N}(E) \sim \mathcal{N}_0(E)$ where $\mathcal{N}_0$ is the IDS of the free case (namely $\nu = 0$).

- Gaps can be labelled by the value the IDS takes on them.
- An example of IDS -
- An example of IDS -
III.4) - $K$-group labels

- If $E$ belongs to a gap $\mathfrak{g}$, the characteristic function $E' \in \mathbb{R} \mapsto \chi(E' \leq E)$ is continuous on the spectrum of $H$. Thus:

$$P_{\mathfrak{g}} = \chi(H \leq E) \text{ is a projection in } \mathcal{A}!$$

- $\mathcal{N}(E) = \mathcal{T}_P(P_{\mathfrak{g}}) \in \mathcal{T}_P^*(K_0(\mathcal{A}))!$

Theorem 2 (Abstract gap labelling theorem)

- $S \subset \Sigma_H$ clopen, $n_S = [\chi_S(H)] \in K_0(\mathcal{A})$. If $S_1 \cap S_2 = \emptyset$ then $n_{S_1 \cup S_2} = n_{S_1} + n_{S_2}$ (additivity).

- Gap labels are invariant under norm continuous variation of $H$ (homotopy invariance).

- For $\lambda \in [0, 1] \mapsto H(\lambda) \in \mathcal{A}$ continuous, if $S(\lambda) \subset \Sigma_H$ clopen, continuous in $\lambda$ with $S(0) = S_1 \cup S_2$, $S(1) = S'_1 \cup S'_2$ and $S_1 \cap S_2 = \emptyset = S'_1 \cap S'_2$ then $n_{S_1} + n_{S_2} = n_{S'_1} + n_{S'_2}$ (conservation of gap labels under band crossings).
IV - Computing Gap Labels


IV.1)- The Main Result

**Theorem 3** If $\mathcal{T}$ is an FPC-tiling in $\mathbb{R}^d$ with Hull $(\Omega, \mathbb{R}^d, \mathcal{T})$, then, for any $\mathbb{R}^d$-invariant probability measure $\mathbb{P}$ on $\Omega$

$$\mathcal{T}^*_\mathbb{P}(K_0(\mathcal{A})) = \int_X d\mathbb{P}_{tr} \ C(X, \mathbb{Z}) .$$

if $\mathcal{A} = C(\Omega) \rtimes \mathbb{R}^d$, $X$ is the canonical transversal and $\mathbb{P}_{tr}$ the transverse measure induced by $\mathbb{P}$.

For $d = 1$ this result follows from the Pimsner & Voiculescu exact sequence (Bellissard, ’92).

For $d = 2$, a double use of the Pimsner & Voiculescu exact sequence provides the result (van Elst, ’95).

For $d \geq 3$ whenever $(\Omega, \mathbb{R}^d, \mathcal{T})$ is Morita equivalent to a $\mathbb{Z}^d$-action, a strategy using spectral sequences led to this theorem for $d = 3$ (Bellissard, Kellendonk, Legrand, ’00).

The theorem has also been proved for all $d$’s recently an independently by (Benamour, Oyono, 2001) and (Kaminker, Putnam, 2001).
IV.2)- Cycles on BOF-Manifolds

A BOF-Manifold $B$ admits a unique finite decomposition into tiles. Each tile inherits the orientation of $B$, thus each face of a tile is also oriented. A positive weight on $B$ is a map $w$ affecting to each positively oriented tile $T$ a positive number $w(T)$ such that

$$w(\overline{T}) = -w(T)$$

and for each oriented face $F$

$$\partial w(F) = \sum_{F \subset \partial T} w(T) = 0 \quad (Kirchhoff's \ law)$$

Thus the set $W(B)$ of positive weights satisfies

$$W(B) = H_d(B, \mathbb{R})^+ \quad d = dim(B)$$

if $H_*(B, \mathbb{R})$ is the homology of the $CW$-complex defined by the tiles and their faces of lower dimensions. The positive cone is defined by the positively oriented tiles.
IV.3)- Invariant Measures


Using

$$\Omega = \lim_{\leftarrow} B_n \quad \Rightarrow \quad H_\ast(\Omega, \mathbb{R}) = \lim_{\leftarrow} H_\ast(B_n, \mathbb{R})$$

**Theorem 4**

1)- The set of $\mathbb{R}^d$-invariant positive bounded measures on $\Omega$ is affinely isomorphic to the positive part of $H_d^\ast(\Omega, \mathbb{R})$.

2)- The direct limit $H^\ast(\Omega, \mathbb{R})$ of the cohomology groups $H^\ast(B_n, \mathbb{R})$ coincides with the de Rham cohomology of $\Omega$ along the orbits of the $\mathbb{R}^d$-action.

3)- The pairing between the invariant measures and $H^d(\Omega, \mathbb{R})$ is given by the Ruelle-Sullivan current

$$\langle \mathbb{P} | [\eta] \rangle = \int_{\Omega} d\mathbb{P} \langle \eta | X_1 \wedge \cdots \wedge X_d \rangle$$

where $X_1, \cdots, X_d$ generate the $\mathbb{R}^d$-action.
IV.4)- K-theory


By the Connes-Thom theorem,

$$K_i(C(\Omega) \rtimes \mathbb{R}^d) \simeq K_{i+d}(C(\Omega)) .$$

For a projector $P \in \mathcal{A} = C(\Omega) \rtimes \mathbb{R}^d$

$$\mathcal{T}_P(P) = \langle \mathbb{P}[\eta_P] \rangle$$

where $[\eta] \in H^d(\Omega, \mathbb{Z})$. In particular $\eta$ can be chosen so that

$$\langle \eta | X_1 \wedge \cdots \wedge X_d \rangle \in C(\Omega, \mathbb{Z})$$

Using the theorem 4, the Main Theorem is proved.