GAP LABELING
THEOREMS

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Main References


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I - Harper’s Model


2D-Crystal Electrons in Magnetic Field

- Perfect *square lattice*, nearest neighbor hoping terms, *uniform magnetic field* $B$ perpendicular to the plane of the lattice
- Translation operators $U_1, U_2$

![Diagram of a 2D lattice with translation operators $U_1$, $U_2$, $U_1^{-1}$, $U_2^{-1}$, and lattice spacing $a$.]

- $a$ = lattice spacing
- $\phi$ = flux through unit cell
2D-Crystal Electrons in Magnetic Field

- Commutation rules \((Rotation\ Algebra)\)
  \[
  U_1 U_2 = e^{2\pi \alpha} U_2 U_1 \quad \alpha = \frac{\phi}{\phi_0} \quad \phi = Ba^2 \quad \phi_0 = \frac{e^2}{h}
  \]

- Kinetic Energy \((Hamiltonian)\)
  \[
  H = t \left( U_1 + U_2 + U_1^{-1} + U_2^{-1} \right)
  \]

- Landau gauge \(\psi(m, n) = e^{2i\pi mk} \varphi(n)\).
  Hence \(H\psi = E\psi\) means
  \[
  \varphi(n + 1) + \varphi(n - 1) + 2 \cos 2\pi(n\alpha - k)\varphi(n) = \frac{E}{t} \varphi(n)
  \]
For $\alpha = p/q$, the following properties hold:

- The spectrum has $q$ nonoverlapping bands, touching only at $E = 0$.
  
  (Bellissard-Simon ’82,..., Avila-Jitomirskaya ’09)

- The spectral gaps are bounded below by $e^{-Cq}$ for some $C > 0$.
  
  (Helffer-Sjöstrand ’86-89, Choi-Elliot-Yui. ’90)
2D-Crystal Electrons in Magnetic Field

For $\alpha \notin \mathbb{Q}$,

- The spectrum is a Cantor set
  
  \textit{(Bellissard-Simon '82,..., Avila-Jitomirskaya '09)}

- The spectrum has zero Lebesgue measure
  
  \textit{(Avron-van Mouche-Simon, '90, ..., Avila-Jitomirskaya '09)}

- The gap edges are Lipshitz continuous as long as they do not close, otherwise they are Hölder with exponent $1/2$
  
  \textit{(Bellissard '94, Avron-van Mouche-Simon, '90, Haagerup et al.)}

- The derivative of gap edges \textit{w.r.t.} $\alpha$ is discontinuous at each rational
  
  \textit{(Wilkinson '84, Rammal '86, Bellissard-Rammal '90)}
The $C^*$-algebra $\mathcal{A}_\alpha$ generated by two unitaries $U_1, U_2$ such that $U_1 U_2 = e^{2i\pi \alpha} U_2 U_1$ is called the rotation algebra (Rieffel '81).

$\mathcal{A}_\alpha$ has a trace defined by

$$\mathcal{T}(U_1^m U_2^n) = \delta_{m,0} \delta_{n,0}$$

$\mathcal{A}_\alpha$ admits two $*$-derivations $\partial_1, \partial_2$ defined by (Connes '82)

$$\partial_i U_j = 2i\pi \delta_{i,j} U_j$$
Rotation Algebra

- Rieffel’s projection $P_R = -f(U_2)U_1 + g(U_2) - U_1^{-1}f(U_2)$

$$T(P_R) = \alpha$$

$$\frac{1}{2i\pi}T(P_R[\partial_1 P_R, \partial_2 P_R]) = 1$$

- If $P \in A_\alpha$ is a projection, then

$$T(P) = n\alpha - [n\alpha]$$

$$n = \text{Ch}(P) = \frac{1}{2i\pi}T(P[\partial_1 P, \partial_2 P]) \in \mathbb{Z}$$

(Rieffel ’81, Pimsner-Voiculescu ’80, Connes ’82)
Gap Labels

• If \( H = U_1 + U_1^{-1} + U_2 + U_2^{-1} \), and if \( E \) belongs to a gap of the spectrum of \( H \), set

\[
P_E = \frac{1}{2i\pi} \oint_{\gamma} \frac{dz}{zI - H}
\]

Then \( P_E \in \mathcal{A}_\alpha \) !!

Hence \( \mathcal{T}(P_E) = n\alpha - [n\alpha] \) for some \( n \in \mathbb{Z} \) !!
Gap Labels

• The spectral projection of the Harper model between any two gaps can be labelled by an integer, using the previous results, (Claro-Wannier ’78)

• This integer corresponds to the quantization of the Hall conductivity in such systems (Thouless-Kohmoto-den Nijs-Nightingale ’82)

Each color corresponds to the integer gap label, for the eigenprojection between the l.h.s and the gap. (Avron-Osadchy-Seiler ’03)
II - Almost Periodic Potentials


Schrödinger’s Operator with Nowhere Dense Spectrum

• 1D-Schrödinger equation with a periodic potential

\[-\frac{d^2 \varphi}{dx^2} + V(x) \varphi(x) = E \varphi(x)\]

\[V(x + 1) = V(x) \quad V \text{ smooth}\]

• Bloch-Floquet: find solutions with \( \varphi(x + 1) = e^{ik} \varphi(x) \), giving

\[E = E_n(k) \quad E_n(k + 2\pi) = E_n(k) \quad n \in \mathbb{N}\]

Band spectrum with gaps at \( k = m\pi \quad m \in \mathbb{Z} \)
Schrödinger’s Operator with Nowhere Dense Spectrum
Add to $V(x)$ a contribution $V_1(x/2)$ with $V_1(x + 1) = V_1(x)$ and $\|V_1\|_\infty < \|V\|_\infty$.

It leads to the opening of new gaps at $k = m\pi/2$ instead.

The size of the new gaps can be controlled.
Schrödinger’s Operator with Nowhere Dense Spectrum
Schrödinger’s Operator with Nowhere Dense Spectrum
Schrödinger’s Operator with Nowhere Dense Spectrum

• Add to $V(x)$ a sequence $V_j(x/2^j)$ with $V_j(x + 1) = V_j(x)$ and

$$
\sum_{j=0}^{\infty} e^{rj} \|V_j\|_{\infty} < \infty
$$

for $r > 0$ large enough.

• It leads to the opening of an *infinite number* of very small gaps at $k = m\pi/2^j$. This leads to a *Cantor spectrum*
Rotation Number

- If \( \varphi(x) \) is a solution of the Schrödinger equation at energy \( E \), the rotation angle is defined by

\[
e^{i\theta(x)} = \frac{\varphi(x) + i\varphi'(x)}{|\varphi(x) + i\varphi'(x)|}
\]

The rotation number \( \rho(E) \) is defined by

\[
\rho(E) = \lim_{L \to \infty} \frac{1}{2\pi L} \int_{-L}^{+L} d\theta(x)
\]
Theorem (Johnson-Moser '82)

If $V$ is a quasiperiodic potential and $E$ belongs to a spectral gap, then $\rho(E)$ belongs to the $\mathbb{Z}$-module generated by the frequencies of $V$. 
III - The Hull of a Hamiltonian


Homogeneity

• Let $H = -\Delta + V$, be a Schrödinger operator on $\mathbb{R}^d$. Let $U(a)$ be the unitary operator on $L^2(\mathbb{R}^d)$ representing the translation by $a \in \mathbb{R}^d$.

• $H$ is called **homogeneous** if the family $\Omega_0(z) = \{U(a)(H - zI)^{-1}U(a)^{-1}; a \in \mathbb{R}^d\}$ is strongly precompact for at least one $z \in \mathbb{C}$ such that $\mathfrak{Im}(z) \neq 0$.

• **Example:** if $V \in L^\infty_{\mathbb{R}}(\mathbb{R}^d)$ then $H$ is homogeneous

• The strong closure of $\Omega_0(z)$ is denoted by $\Omega$: it is a compact metrizable set, independent of $z$ modulo homeomorphisms called the **Hull** of $H$. 
Homogeneity

- The translation group $\mathbb{R}^d$ acts on $\Omega$ by homeomorphisms $t$ and $(\Omega, \mathbb{R}^d)$ is a topological dynamical system.

- Let $H$ be homogeneous. Then, each $\omega \in \Omega$ defines a selfadjoint operator $H_\omega$ on $L^2(\mathbb{R}^d)$ through taking the strong resolvent limit. Then
  
  - $\omega \in \Omega \rightarrow (H_\omega - zI)^{-1}$ is strongly continuous (for $z \notin \mathbb{R}$)
  - $U(a)H_\omega U(a)^{-1} = H_{t^a\omega}$ (covariance)

- $H_\omega = -\Delta + V_\omega$ where, if $V$ is continuous, $V_\omega(x) = v(t^{-x}\omega)$ with $v \in C(\Omega)$. 
$\mathcal{C}^*$-algebra

The crossed product algebra $\mathcal{A} = C(\Omega) \rtimes \mathbb{R}^d$ is constructed as follows. Let $\mathcal{A}_0 = C_c(\Omega \times \mathbb{R}^d)$

- **Product:** if $A, B \in \mathcal{A}_0$ then

$$AB(\omega, x) = \int_{\mathbb{R}^d} A(\omega, y) B(t^{-y} \omega, x - y) \, d^dy$$

- **Adjoint:** if $A \in \mathcal{A}_0$ then

$$A^*(\omega, x) = A(t^{-x} \omega, -x)$$

- **Left Regular Representation:** if $A \in \mathcal{A}_0$ and if $\psi \in L^2(\mathbb{R}^d)$ then

$$\pi_\omega(A) \psi(x) = \int_{\mathbb{R}^d} A(t^{-x} \omega, y - x) \, \psi(y) \, d^dy$$
\(C^*-\text{algebra}\)

- **\(C^*\text{-norm:}\)** if \(A \in \mathcal{A}_0\) then

\[
\|A\| = \sup_{\omega \in \Omega} \|\pi_\omega(A)\|
\]

- **\(C^*\text{-algebra:}\)** \(\mathcal{A} = C(\Omega) \rtimes \mathbb{R}^d\) is the completion of \(\mathcal{A}_0\) w.r.t. the norm \(\| \cdot \|\)

- **Theorem:** (Bellissard ’86, using Woronowicz, Baaj, Doplicher et al, Georgescu ’02)

If \(H\) is homogeneous, it is affiliated to \(\mathcal{A}\):

*namely, there is a \(*\text{-homomorphism}*

\[
f \in C_0(\mathbb{R}) \mapsto f(H) \in \mathcal{A}
\]

such that \(\pi_\omega(f(H)) = f(H_\omega)\) for all \(\omega \in \Omega\).
Energy Spectrum

• A-spectrum: \( \text{Sp}_A(H) \) is the complement of the domain of holomorphy of \( R_H(z) = (H - zI)^{-1} \in A \).

• Gaps: Since \( H \) is selfadjoint, \( \text{Sp}_A(H) \subset \mathbb{R} \) and is closed. A gap is a connected component of its complement \( \mathbb{R} \setminus \text{Sp}_A(H) \).

• Proposition: (Bellissard '86)
  
  – \( \text{Sp}_A(H) \) is the union over \( \omega \) of \( \text{Sp}(H_\omega) \)
  
  – If the orbit of \( \omega \in \Omega \) is dense then \( \text{Sp}_A(H) = \text{Sp}(H_\omega) \)
  
  – If there is a periodic orbit in \( \Omega \) then \( \text{Sp}_A(H) \) cannot be nowhere dense
Calculus

- **Trace**: let $\mathbb{P}$ be an ergodic, $\mathbb{R}^d$-invariant probability measure on $\Omega$. A *trace* on $\mathcal{A}$ is defined by

$$
\mathcal{T}_\mathbb{P}(A) = \int_\Omega A(\omega, 0) \, d\mathbb{P}(\omega) \quad A \in \mathcal{A}_0
$$

- **Trace per Unit Volume**: if $\Lambda$ are cubes centered at the origin and if $\chi_\Lambda$ is the characteristic function of $\Lambda$, then, Birkhoff’s ergodic theorem leads to

$$
\mathcal{T}_\mathbb{P}(A) = \lim_{\Lambda \uparrow \mathbb{R}^d} \frac{1}{|\Lambda|} \text{Tr} \left( \pi_\omega(A) \chi_\Lambda \right) \quad \mathbb{P}\text{-almost all } \omega
$$
Calculus

• **Dual Action** (Connes, Takai-Takesaki) \( \eta_k(A)(\omega, x) = e^{ik \cdot x} A(\omega, x) \), with \( k \in \mathbb{R}^d \), defines a norm pointwise continuous *d*-parameter group of \(*\)-automorphisms.

• **Differential Structure:** The dual action is generated by the following \(*\)-derivations

\[
\partial_j A(\omega, x) = ix_j A(\omega, x) \quad \text{for} \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d
\]

• **Position Operators:** let \( X = (X_1, \ldots, X_d) \) be the operators on \( L^2(\mathbb{R}^d) \) defined by \( X_j \psi(x) = x_j \psi(x) \). Then

\[
\pi_\omega \left( \partial_j A \right) = i \left[ X_j, \pi_\omega(A) \right]
\]
The Integrated Density of States

- **Integrated Density of States:** The restriction \( H_{\omega,\Lambda} \) of \( H_\omega \) to a bounded domain \( \Lambda \) is elliptic. Hence its spectrum is discrete. The *IDS* is defined by

\[
N_{P}(E) = \lim_{\Lambda \uparrow \mathbb{R}^d |\Lambda|} \frac{1}{\# \{ \text{eigenvalue of } H_{\omega,\Lambda} \leq E \}} \quad P\text{-almost all } \omega
\]

- **Properties:**
  - \( N_{P} \) is nondecreasing w.r.t. \( E \)
  - \( N_{P}(E) = 0 \) for \( E < \inf \text{SpH} \)
  - \( N_{P}(E) \sim E^{d/2} \) as \( E \to +\infty \)
  - \( N_{P} \) is constant on spectral gaps.
The Integrated Density of States

An example of IDS
The Integrated Density of States

IDS for a Rudin-Shapiro potential
(Montalbana et al. '07)
**Gap Labels**

- **Shubin’s Formula:** (Shubin ’76, Bellissard ’86)
  
  \[ N_{\mathbb{P}}(E) = \mathcal{T}_{\mathbb{P}}(P_{E}) \quad P_{E} = \chi_{(-\infty,E]}(H) \]

- **Spectral Projections:**
  - If \( E \in \text{Sp}_{A}(H) \) then \( P_{E} \) is a projector in \( L^{\infty}(A,\mathcal{T}_{\mathbb{P}}) \)
  - *If E is in a gap*, then \( P_{E} \in A \)
  - \( P_{E} \) does not change as \( E \) moves in the *same gap*

- If \( E \) is in a gap \( \mathcal{T}_{\mathbb{P}}(P_{E}) \) depends only upon the equivalent class of \( P_{E} \) in \( K_{0}(A) \)
Gap Labels

- $\mathcal{T}_\mathbb{P}$ induces a group homomorphism $\mathcal{T}_* : K_0(\mathcal{A}) \rightarrow \mathbb{R}$. Since $K_0(\mathcal{A})$ is countable its image, the group of gap labels, is a countable subgroup of $\mathbb{R}$.

- Gap Labeling Theorem:
  - the values of the IDS on gaps belongs to the image of $K_0(\mathcal{A})$ under the trace
  - if $g_0 < g_1$ are two gaps the difference $N_{\mathbb{P}}(g_1) - N_{\mathbb{P}}(g_0)$ is also a gap labels.
  - If $H = H(s)$ depends continuously (norm resolvent) on a parameter $s$ then the gap label of a gap does not change as long as the gap does not close.
Gap Labels

- **Sum Rule:** let $H = H(s)$ depends continuously (norm resolvent) on a parameter $s \in [0, 1]$

$$m_0 + m_1 = n_0 + n_1$$
IV  - Computing the Gap Labels

Quasiperiodic Potentials

- For $n > d$ integers, let $\kappa$ be an $n \times d$ matrix of rank $d$ with real coefficients.
  $\kappa$ is *irrational* whenever $\text{Im}(\kappa) \cap \mathbb{Z}^n = \{0\}$.

- For $V \in C(\mathbb{T}^n)$ let $V(x) = V(-\kappa x)$ for $x \in \mathbb{R}^d$.
  Then $V$ is *quasiperiodic*. Conversely every quasiperiodic function on $\mathbb{R}^d$ can be written in this way.

- **Theorem**: (Johnson-Moser ’82, Bellissard ’93 using Connes Index Theorem for foliations)
  
  - If $H = -\Delta + V$ then $\Omega = \mathbb{T}^n$
  
  - $\mathbb{R}^d$-action: $\tau^{-a} \omega = \omega - \kappa a$ is uniquely ergodic
  
  - Then $V_\omega(x) = V(\omega - \kappa x)$ if $\omega \in \mathbb{T}^n$
  
  - The group of gap labels is the $\mathbb{Z}$-module spanned by $\det(\beta)$’s where $\beta$ runs through the set of submatrices of maximal rank of $\kappa$
Quasiperiodic Potentials

In 1D, more can be said

- Let \( \psi(\omega, x) \) be the \textit{unique solution} of the Schrödinger equation

\[
-\psi'' + V_\omega \psi = E \psi \quad \lim_{x \to +\infty} \psi(x) = 0 \quad \psi(0) = 1
\]

- It defines a curve in the complex plane through

\[
\Phi(\omega, x) = \psi(\omega, x) + \frac{1}{\sqrt{E}} \frac{d\psi(\omega, x)}{dx} \neq 0
\]

- The covariance property shows that

\[
F(\omega, x) = \Phi(\omega, x)^{-1} \frac{d\Phi(\omega, x)}{dx} \implies F(\omega, x) = F(\omega - \kappa x, 0)
\]
Quasiperiodic Potentials

- **Sturm-Liouville:** The number of eigenvalues smaller than or equal to $E$ of the restriction to $[-L, L]$ of $H_\omega$ is given by

  $$\frac{1}{2i\pi} \int_{-L}^{+L} \Phi(\omega, x)^{-1} \frac{d\Phi(\omega, x)}{dx} \, dx$$

  (rotation number)

- Hence, using the covariance and Birkhoff's ergodic theorem, the IDS becomes

  $$\mathcal{T}(P_E) = N(E) = \frac{1}{2i\pi} \int_{\mathbb{T}^n} \Phi(\omega, 0)^{-1} \frac{d\Phi(\omega, 0)}{dx} \, d\omega$$

- **Comments:** (i) this gives a proof of the Johnson-Moser result, (ii) it is a special case of the Connes index formula for foliations.
Atomic Potentials with Finite Local Complexity

- Let $\mathcal{L} \subset \mathbb{R}^d$ be uniformly discrete:
  \[ \inf\{|x - y| ; x \neq y, x, y \in \mathcal{L}\} > 0 \]

- $V$ is an atomic potential on $\mathcal{L}$ if
  \[ V(x) = \sum_{y \in \mathcal{L}} v(x - y) \quad v \in L^1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d) \]

- A patch is a finite subset of $\mathcal{L}$, modulo translations. $\mathcal{L}$ has finite local complexity if, for any $R > 0$ the set of patches of diameter $R > 0$ is finite.
Atomic Potentials with Finite Local Complexity

• **Theorem:** Let $V$ be an atomic potential with finite local complexity and let $H = -\Delta + V$
  
  – The Hull of $H$ is a compact space, foliated by the action of $\mathbb{R}^d$, with a completely discontinuous transversal  
  
  – The set of gap labels is the $\mathbb{Z}$-module generated by the occurrence probabilities of patches w.r.t. the probability $P$ on the Hull

  (Bellissard ‘92, Kaminker-Putnam ‘01, Bellissard-Benedetti-Gambaudo ‘01–’06, Oyono-Oyono & Benameur ‘01–’07)

• **Examples:**

  – If $d = 1$ and $\mathcal{L}$ is the *Fibonacci chain*, the set of gap labels is $\mathbb{Z} + \sigma \mathbb{Z}$ where $\sigma = (\sqrt{5} - 1)/2$

  – Same results if $\mathcal{L}$ is the *Penrose lattice* or an *icosahedral quasicrystal* in 3D.
It is time for coffee!