The

TRANSVERSE GEOMETRY

of

TLING SPACES

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Collaborations

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Main References


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2. Spectral Triple
3. The Pearson Laplacian
4. Open Problems
I - Tilings and their Transversal
The Fibonacci Tiling

The Fibonacci Substitution
The Fibonacci Tiling

\[ a_0 \]

\[ a_1 \]

\[ a_2 \]

\[ a_3 \]

\[ a \]

\[ ab \]

\[ aba \]

\[ abaab \]
The Fibonacci Tiling

cut-and-project version of the Fibonacci sequence

\[ \text{Slope} = \frac{\sqrt{5} - 1}{2} = \sigma \]
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Symmetry around (.5,.5)

window
unit cell

$E_{\perp}$

$E_{||}$

$a_1, a_2, a_3$
The Octagonal Tiling

Octagonal Lattice

$\mathbb{Z}^4 \rightarrow \mathbb{R}^2$
The Octagonal Tiling

Octagonal Lattice

$Z^4 \rightarrow \mathbb{R}^2$

Substitution
The Octagonal Tiling

Octagonal Lattice

$\mathbb{Z}^4 \rightarrow \mathbb{R}^2$

The Transversal or Window
The Octagonal Tiling

Octagonal Lattice

$\mathbb{Z}^4 \rightarrow \mathbb{R}^2$

Local Environments
Inverse Limit

Let $\mathcal{P}_R$ be the set of patches of radius $R$, modulo translation.

The tiling has finite local complexity (FLC), if and only if $\mathcal{P}_R$ is a finite set for all $R$. In particular $R \to \mathcal{P}_R$ is locally constant and non-decreasing. Thus there is a sequence $R_0 = 0 < R_1 < \cdots < R_n < \cdots$ with $R_n \to \infty$ such that $\mathcal{P}_R = \mathcal{P}_n$ for $R_n \leq R < R_{n+1}$.
There is a restriction map $\pi : P_{n+1} \to P_n$. Then the transversal is defined by the inverse limit

$$\Xi = \lim_{\leftarrow \pi} P_n$$
Since all the $P_n$'s are finite set, $\Xi$ is a Cantor set.

A point of $\Xi$ is an infinite sequence $\xi = (p_n)_{n=0}^{\infty}$ of compatible patches, so it defines a unique tiling.

This inverse limit can be represented by a rooted tree.
Rooted Tree

For the *Fibonacci sequence* this gives

**The Fibonacci Tree**
II - Spectral Triples
A spectral triple for a C*-algebra $\mathcal{A}$ is a family $X = (\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{H}$ is a Hilbert space, $D$ and unbounded operator on $\mathcal{H}$ such that
A \textit{spectral triple} for a C\(^*\)-algebra \(\mathcal{A}\) is a family \(X = (\mathcal{A}, \mathcal{H}, D)\) where \(\mathcal{H}\) is a Hilbert space, \(D\) and unbounded operator on \(\mathcal{H}\) such that

- there is a (faithful) representation \(\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})\)
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- the set $C^1(X)$ of elements $a \in \mathcal{A}$ leaving the domain of $D$ invariant and such that $\|[D, \pi(a)]\| < \infty$, is dense in $\mathcal{A}$
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- the set $C^1(X)$ of elements $a \in \mathcal{A}$ leaving the domain of $D$ invariant and such that $\|[D, \pi(a)]\| < \infty$, is dense in $\mathcal{A}$

**Proposition:** Then $C^1(X)$ is a dense ∗-subalgebra of $\mathcal{A}$, invariant under the holomorphic functional calculus.
Example

Let $M$ be a spin$^c$ Riemannian manifold, $\mathcal{A} = C(M)$, $\mathcal{H}$ the space of $L^2$-sections of the spin bundle and $D$ the corresponding Dirac operator, where $\mathcal{A}$ acts by pointwise multiplication.
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Actually $\|[D, f]\| = \|\nabla f\|_\infty = \|f\|_{C_{\text{Lip}}} and C^1(X) = \text{Lip}(M)$.
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Actually $\|[D, f]\| = \|\nabla f\|_{L^\infty} = \|f\|_{C^{\text{Lip}}} \text{ and } C^1(X) = \text{Lip}(M)$.

Hence the algebra $\mathcal{A}$ encodes the *space*, the Dirac operator $D$ encodes the *metric*. $\mathcal{H}$ is needed to define $D$. 
Ultrametric on $\Xi$

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**Theorem, (Michon ‘84)** If $\xi, \eta \in \Xi$ let $\xi \land \eta$ be the least common ancestor of the path $\xi$ and $\eta$. Then $d_\kappa(\xi, \eta) = \kappa(\xi \land \eta)$ defines an ultrametric on $\Xi$. 
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Then $\kappa(p)$ is the diameter of the set of tilings compatible with $p$. 
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Each ultrametric on $\Xi$ can be obtained in such a way through a rooted tree defined from the metric.
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• If $p$ is a patch, take $\kappa(p)$ to be the maximum potential energy difference at the origin, produced by atoms outside $p$ on all tilings of $\Xi$ compatible with $p$. 
The Pearson-Palmer Spectral Triple

Given $p$ a patch, let $\Xi(p)$ be the set of all tilings in $\Xi$ compatible with $p$ at the origin. The family $(\Xi(p))_{p \in \mathcal{P}}$ is a basis of clopen set for the topology of $\Xi$. 
The Pearson-Palmer Spectral Triple

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$$\text{diam } \mathcal{P} = \max\{\kappa(p) ; p \in \mathcal{P}\}$$

An infinite sequence $(\mathcal{P}_n)_{n\in \mathbb{N}}$ of clopen cover is called *resolving* if $\lim_{n\to \infty} \text{diam } \mathcal{P}_n = 0$. 
The Pearson-Palmer Spectral Triple

- Algebra: $\mathcal{A} = C(\Xi)$,
The Pearson-Palmer Spectral Triple

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- **Hilbert Space:** \( \mathcal{H} = \bigoplus_{n \in \mathbb{N}} \ell^2(\mathcal{P}_n) \otimes \mathbb{C}^2 \), with \( (\mathcal{P}_n)_{n \in \mathbb{N}} \) a resolving sequence of clopen covers.
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- **Dirac Operator:** for $\psi \in \mathcal{H}$
  $$ (D\psi)(p) = \frac{1}{\kappa(p)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \psi(p). $$
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- **Representation:** for each choice \( \tau \) and \( f \in C(\Xi) \)
  \[
  (\pi_\tau(f)\psi)(p) = \begin{bmatrix} f(\xi_p) & 0 \\ 0 & f(\eta_p) \end{bmatrix} \psi(p).
  \]
The $\zeta$-function is defined by

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If so, the abscissa of convergence, defined by $s_0 = \inf\{s > 0 ; \zeta(s) < \infty\}$ satisfies

$$s_0 \geq \dim_H(\Xi)$$
**ζ-function**

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\]

There exists a (non unique) resolving sequence of clopen covers \( (P_n)_{n \in \mathbb{N}} \), called a Hausdorff sequence, such that \( s_0 = \dim_H(Ξ) \).
The Connes State

The Connes state is defined by

$$ T(f) = \lim_{s \to s_0} \frac{1}{\zeta(s)} \text{Tr} \left( \frac{1}{|D|^s} \pi_\tau(f) \right), \quad f \in \mathbb{C}(\Xi) $$
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**Theorem:** If \((\Xi, d_\kappa)\) has finite Hausdorff dimension and if \((P_n)_{n \in \mathbb{N}}\) is a Hausdorff sequence, the Connes state exists if and only if \(\Xi\) has a finite nonzero Hausdorff measure.
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**Theorem:** If \((\Xi, d_\kappa)\) has finite Hausdorff dimension and if \((\mathcal{P}_n)_{n \in \mathbb{N}}\) is a Hausdorff sequence, the Connes state exists if and only if \(\Xi\) has a finite nonzero Hausdorff measure.

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If so, \(\mathcal{T}\) coincides with the normalized Hausdorff measure on \(\Xi\).
III - The Pearson Laplacian
Directional Derivative, Tangent Space

If \( \tau(p) = (\xi_p, \eta_p) \) then

\[
[D, \pi_{\tau}(f)] \psi(p) = \frac{f(\xi_p) - f(\eta_p)}{d(\xi_p, \eta_p)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \psi(p)
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Directional Derivative, Tangent Space

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- $\tau(p)$ can be interpreted as a coarse grained version of a unit tangent vector at $p$. 
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The commutator with the Dirac operator is a coarse grained version of a *directional derivative*. In particular

- $\tau(p)$ can be interpreted as a coarse grained version of a *unit tangent vector* at $p$.
- the set $\Upsilon$ of all possible choices, can be seen as the set of *sections of the tangent sphere bundle*. 

Directional Derivative, Tangent Space

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The commutator with the Dirac operator is a coarse grained version of a directional derivative. In particular

- $\tau(p)$ can be interpreted as a coarse grained version of a unit tangent vector at $p$.
- the set $\Upsilon$ of all possible choices, can be seen as the set of sections of the tangent sphere bundle.
- $[D, \pi_\tau(f)]$ could be written as $\nabla_\tau f$. 
Choice Averaging

- The *choice space* $\Upsilon$ is given by $\prod_p \Upsilon(p)$ where $\Upsilon(p)$ is a clopen subset of $\Xi(p) \times \Xi(p)$. 
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- Let $\nu_p$ be the probability measure on $\Upsilon(p)$ induced by the Hausdorff measure $\mu_H \otimes \mu_H$ on $\Xi(p) \times \Xi(p)$.
Choice Averaging

• The choice space $\mathcal{Y}$ is given by $\prod_p \mathcal{Y}(p)$ where $\mathcal{Y}(p)$ is a clopen subset of $\Xi(p) \times \Xi(p)$.

• Let $\nu_p$ be the probability measure on $\mathcal{Y}(p)$ induced by the Hausdorff measure $\mu_H \otimes \mu_H$ on $\Xi(p) \times \Xi(p)$.

• This leads to the probability

$$\nu = \bigotimes_p \nu_p$$
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- This leads to the probability

$$\nu = \bigotimes_p \nu_p$$

Hence $\nu_p$ can be interpreted as the average over the tangent unit sphere at $p$. 
The Pearson quadratic form is defined by (if \( f, g \in C(\Xi) \))

\[
Q_s(f, g) = \int_{\Upsilon} d\nu(\tau) \; \text{Tr} \left( \frac{1}{|D|} [D, \pi_\tau(f)]^* [D, \pi_\tau(g)] \right)
\]
The Pearson Quadratic Form

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**Theorem:** If \((\Xi, d_\kappa)\) has positive finite Hausdorff measure, for each \( s \in \mathbb{R} \), the quadratic forms \( Q_s \) is densely defined, closable in \( L^2(X, \mu_H) \) and is a Dirichlet form.
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**Theorem:** If $(\Xi, d_\kappa)$ has positive finite Hausdorff measure, for each $s \in \mathbb{R}$, the quadratic forms $Q_s$ is densely defined, closable in $L^2(X, \mu_H)$ and is a Dirichlet form.

The corresponding positive operator $\Delta_s$ has pure point spectrum. It is bounded if and only if $s > \dim_H(\Xi) + 2$ and has compact resolvent otherwise.
The Pearson Quadratic Form

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$$Q_s(f, g) = \int_{\gamma} d\nu(\tau) \text{Tr} \left( \frac{1}{|D|^s} [D, \pi_\tau(f)]^* [D, \pi_\tau(g)] \right)$$

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The corresponding positive operator $\Delta_s$ has pure point spectrum. It is bounded if and only if $s > \dim_H(\Xi) + 2$ and has compact resolvent otherwise.

The eigenspaces are common to all $s$’s and can be explicitly computed.
Jump Process

$\Delta_s$ generates a Markov semigroup, thus a stochastic process $(X_t)_{t \geq 0}$ where the $X_t$'s takes on values in $\Xi$. 
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Given a patch $p$, its spine is the set of vertices located along the finite path joining the root to $p$. The vine $\mathcal{V}(p)$ of $p$ is the set of patches, not in the spine, which are children of one vertex of the spine.
The vine of a vertex $v$

The vine of $p$
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If \(\chi_p\) is the characteristic function of \(\Xi(p)\), the Pearson operator acts as

\[
\Delta_s \chi_p = \sum_{q \in \mathcal{V}(p)} M(p, q)(\chi_q - \chi_p)
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where \(M(p, q) > 0\) represents the probability rate (per unit time) for \(X_t\) to jump from \(\Xi(p)\) to \(\Xi(q)\).
Jump process from \( p \) to \( q \)

Jump process from \( v \) to \( w \)
Jump Process

Concretely, if $\hat{q}$ denotes the father of $q$ (which belongs to the spine)

$$M(p, q) = 2\kappa(\hat{q})^{s-2} \frac{\mu_p}{Z_{\hat{q}}} \quad \mu_p = \mu_H(\Xi(p))$$

where $Z_{\hat{q}}$ is the normalization constant for the measure $\nu_{\hat{q}}$ on the set of choices at $\hat{q}$, namely

$$Z_{\hat{q}} = \sum_{q' \neq q'' \in \text{Ch}(\hat{q})} \mu_{q'} \mu_{q''}$$

where $\text{Ch}(\hat{q})$ denotes the set of children of $\hat{q}$. 
Thanks for Listening!