QUANTUM COMPUTING

an introduction

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A FAST GROWING SUBJECT:

elements for a history
Feynman’s proposal:

He suggested in 1982 that quantum computers might have fundamentally more powerful computational abilities than conventional ones (basing his conjecture on the extreme difficulty encountered in computing the result of quantum mechanical processes on conventional computers, in marked contrast to the ease with which Nature computes the same results), a suggestion which has been followed up by fits and starts, and has recently led to the conclusion that either quantum mechanics is wrong in some respect, or else a quantum mechanical computer can make factoring integers "easy", destroying the entire existing edifice of public key cryptography, the current proposed basis for the electronic community of the future.

Richard P. Feynman.
Quantum mechanical computers.
Optics News,
Deutsch’s computer:

**David Deutsch.**
Quantum theory, the Church-Turing Principle and universal quantum computer.

David Deutsch.
Conditional quantum dynamics and logic gates.
Shor’s algorithm:

This algorithm shows that a quantum computer can factorize integers into primes in polynomial time.

Peter W. Shor.
Algorithm for quantum computation: discrete logarithms and factoring
*Proc. 35th Annual Symposium on Foundation of Computer Science*,
CSS error-correcting code:

Good quantum error-correcting codes exist

A. M. Steane
Error-correcting codes in quantum theory
Topological error-correcting codes:

Alex Yu. Kitaev.
Fault-tolerant quantum computation by anyons
Books, books, books...
And much more at...

http://www.math.gatech.edu/~jeanbel/4803/

reports
articles,
books,
journals,

list of laboratories,
list of courses,
list of conferences,
QUBITS:

a unit of quantum information

10110101
Qubits:

- George BOOLE (1815-1864) used only two characters to code logical operations.
Qubits:

- John von NEUMANN (1903-1957) developed the concept of programming using also binary system to code all information.
Qubits:

- Claude E. SHANNON « A Mathematical Theory of Communication » (1948)
- Information theory
- Unit of information \textit{bit}
Qubits:

\[ |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

quantizing

\[ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

canonical basis in \( \mathbb{C}^2 \)

1-qubit
Qubits: 1 general qubit

\[ |\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = a |0\rangle + b |1\rangle \]

\[ \langle \psi | = (a^*, b^*) = a^* \langle 0| + b^* \langle 1| \]

*Dirac’s bra and ket in \( \mathbb{C}^2 \) and its dual*
Qubits:

1 general qubit

\[ |q_i\rangle = \begin{pmatrix} a_i \\ b_i \end{pmatrix} = a_i |0\rangle + b_i |1\rangle \]

\[ \langle q_1 | q_2 \rangle = a_1^* a_2 + b_1^* b_2 \]

inner product in \( \mathbb{C}^2 \) using Dirac’s notations
Qubits:

\[ |\psi_1\rangle \langle \psi_2| = \begin{pmatrix} a_1 a_2^* & a_1 b_2^* \\ b_1 a_2^* & b_1 b_2^* \end{pmatrix} \]

\[ \text{Tr} (|\psi_1\rangle \langle \psi_2|) = \langle \psi_2 | \psi_1 \rangle \]

using Dirac’s bra-ket’s
Qubits:

1 general qubit

\[ |\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = a |0\rangle + b |1\rangle \]

\[ \langle \psi | \psi \rangle = |a|^2 + |b|^2 = 1 \]

one qubit = element of the unit sphere in \(\mathbb{C}^2\)
Qubits:

1 general qubit

\[ | \Psi \rangle = \begin{pmatrix} a \\ b \end{pmatrix} = a |0\rangle + b |1\rangle \]

\[ |a|^2 = \text{Prob} (x=0) = |\langle \Psi |0\rangle|^2 \]

\[ |b|^2 = \text{Prob} (x=1) = |\langle \Psi |1\rangle|^2 \]

Born’s interpretation of a qubit
Qubits:

\[ |\psi\rangle \times |\psi\rangle = \text{Projection on } |\psi\rangle \]

\[ p_i \geq 0, \quad \sum_i p_i = 1 \]

\[ |\psi\rangle = \sum_i p_i |\psi_i\rangle \times |\psi_i\rangle \]

\[ \geq 0, \quad \text{Tr}(\square) = 1 \]

\textit{statistical mixtures of states: density matrices}
Qubits:

1 qubit: mixtures

Pauli matrices generate $M_2(\mathbb{C})$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Qubits:  

1 qubit: mixtures

\[ \rho \geq 0 \text{, } Tr(\rho) = 1 \]

\[ \rho = (1 + a_x X + a_y Y + a_z Z)/2 \]

\[ a_x^2 + a_y^2 + a_z^2 \leq 1 \]

density matrices:

the **Bloch** ball
Qubits:

1 qubit: Bloch’s ball
Qubits:

general $N$-qubits states

$01001 \rightarrow |01001\rangle = |0 \rangle \otimes |1 \rangle \otimes |0 \rangle \otimes |0 \rangle \otimes |1 \rangle$

quantizing \hspace{5cm} tensor basis in $\mathbb{C}^{2^n}$
Qubits: general N-qubits states

\[ |\psi\rangle = \sum a(x_1, \ldots, x_N) |x_1 \ldots x_N\rangle \]

\[ \sum |a(x_1, \ldots, x_N)|^2 = 1 \]

*entanglement:* an N-qubit state is NOT a tensor product
Qubits:

general \( N \)-qubits states

\[
\begin{align*}
|\Psi_{00}\rangle &= (|00\rangle + |11\rangle)/\sqrt{2} \\
|\Psi_{01}\rangle &= (|01\rangle + |10\rangle)/\sqrt{2} \\
|\Psi_{10}\rangle &= (|00\rangle - |11\rangle)/\sqrt{2} \\
|\Psi_{01}\rangle &= (|01\rangle - |10\rangle)/\sqrt{2}
\end{align*}
\]

entanglement: Bell’s states
QUANTUM GATES:

computing in quantum world
Quantum gates:

1-qubit gates

$|x> \xrightarrow{U} U|x>$

$U$ is unitary in $M_2(\mathbb{C})$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Pauli basis in $M_2(\mathbb{C})$
Quantum gates:

1-qubit gates

\[ H = 2^{-1/2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \]

Hadamard, phase and $\pi/8$ gates
Quantum gates:

\[ |x_1 \rangle \]
\[ |x_2 \rangle \]
\[ |x_3 \rangle = |x_1 \rangle |x_2 \rangle \ldots |x_N \rangle \]
\[ U |x_1 \rangle |x_2 \rangle \ldots |x_N \rangle \]

\[ U \text{ is unitary in } M_{2^N} (\mathbb{C}) \]
Quantum gates: controlled gates

$U$ is unitary in $M_2(\mathbb{C})$
Quantum gates:

flipping a bit in a controlled way: the CNOT gate

$U=X$
Quantum gates:

Controlling gates

Flipping bits in a controlled way
Quantum gates:

controlled gates

flipping bits in a controlled way

The Toffoli gate
QUANTUM CIRCUITS:

computing in quantum world
Quantum circuits: measurement

- Device that produces a value of the $\text{bit } x$
- The part of the state corresponding to this line is lost.
Quantum circuits: teleportation

\[ |00> \]

\[ |000> \]
Quantum circuits: teleportation

\[ |00\rangle \quad H \quad |\rangle \quad \text{Ignored}

\[ |00\rangle \quad |00\rangle \quad X \quad Z \quad |11\rangle \quad \text{Ignored}

\[ \sqrt{2} |00\rangle + |11\rangle \]
Quantum circuits: teleportation

$|00\rangle$ → $H$

$|00\rangle$ → $X$

$|00\rangle$ → $Z$

$\sqrt{2} |x0\rangle + |x(1-x)\rangle$

$|0\rangle$
Quantum circuits: teleportation

\[ |\psi 00\rangle = \left( |0x0\rangle + (-)^x |1x0\rangle + |0 (1-x)1\rangle + (-)^x |1 (1-x)1\rangle \right) / \sqrt{2} \]

\[ H \]

\[ X \]

\[ Z \]
Quantum circuits: \textit{teleportation}

\[\begin{align*}
\left|\begin{array}{c}
|00\rangle
\end{array}\right> & \xrightarrow{H} \left|\begin{array}{c}
|0\rangle + (-)^x |1\rangle
\end{array}\right> \\
\left|\begin{array}{c}
|00\rangle
\end{array}\right> & \xrightarrow{X} \left|\begin{array}{c}
|x\rangle + (-)^x |x\rangle
\end{array}\right> \\
\left|\begin{array}{c}
|00\rangle
\end{array}\right> & \xrightarrow{Z} \left|\begin{array}{c}
(1-x)x + (-)^x |1-x\rangle
\end{array}\right>
\end{align*}\]
Quantum circuits: teleportation

\[ |0 \rangle \rightarrow H |0 \rangle \rightarrow X |x \rangle \rightarrow Z |0 \rangle \rightarrow 2 \text{ tensor product with } |x \rangle \]

\[ |0_0 \rangle \rightarrow \{ H |0 \rangle \rightarrow X |0 \rangle \rightarrow Z |1 \rangle \rightarrow \text{ teleportation} \]

\[ (|0 \rangle + |1 \rangle + |0 (1-x) \rangle + |1 (1-x) \rangle) \otimes |x \rangle \]
Quantum circuits: teleportation

\[ |\psi>| \rightarrow |\psi_0>| \rightarrow H |\psi_0>| \rightarrow H |\psi_0>| \rightarrow X Z |\psi_0>| \rightarrow |\psi>| \otimes |x> \]

\[ \frac{|00>+|11>+|01>+|10>}{\sqrt{2}} \otimes |x> \]
QUANTUM COMPUTERS:

machines and laws of Physics
Computers:

*Computers are machines obeying to laws of Physics:*

- Non equilibrium Thermodynamics,
- Electromagnetism
- Quantum Mechanics
Computers:

Second Law of Thermodynamics

• Over time, the information contained in an isolated system can only be destroyed

• Equivalently, its entropy can only increase
Computers are machines producing information:

• Coding, transmission, reconstruction
• Computation,
• Cryptography
Computers:

- Coding theory uses redundancy to transmit binary bits of information

0  
coding

1
Computers:

- Coding theory uses redundancy to transmit binary bits of information

0 $\rightarrow$ 000

coding

1 $\rightarrow$ 111
Computers:

- Coding theory uses redundancy to transmit binary bits of information.

\[
\begin{align*}
0 &\rightarrow 000 \\
1 &\rightarrow 111
\end{align*}
\]
Computers:

Coding theory uses redundancy to transmit binary bits of information.

0 → 000
1 → 111

Transmission errors (2nd Law)

010 → 110
Computers:

- Coding theory uses redundancy to transmit binary bits of information.
Computers:

- Coding theory uses redundancy to transmit binary bits of information.
Computers:

Principles of Quantum Mechanics

- States (*pure*) of a system are given by units vectors in a Hilbert space $\mathcal{H}$

- Observables are selfadjoint operators on $\mathcal{H}$ (Hamiltonian $H$, Angular momentum $L$, etc)
Computers:

Principles of Quantum Mechanics

- Quantum Physics is fundamentally *probabilistic*:
  - theory can only predicts the probability distribution of a possible state or of the values of an observable
  - it cannot predict the actual value observed in experiment.
Computers:

Principles of Quantum Mechanics

Where one specific electron shows up is unpredictable. But the distribution of images of many electrons can be predicted.
Computers:

**Principles of Quantum Mechanics**

- $|\langle \psi | \phi \rangle|^2$ represents the probability that $|\phi\rangle$ is in the state $|\psi\rangle$.

- Measurement of $A$ in a state $|\psi\rangle$ is given by

$$\langle f(A) \rangle = \langle \psi | f(A) | \psi \rangle = \int d\mu_{\psi}(a) f(a)$$

where $\mu_{\psi}$ is the *probability distribution* for possible values of $A$. 
Computers:

Principles of Quantum Mechanics

• Time evolution is given by the Schrödinger equation
  \[ i\frac{d\left|\psi\right>}{dt} = H \left|\psi\right> \]
  \[ H = H^* . \]

• Time evolution is given by the unitary operator \( e^{-itH} \) \( \mapsto \) no loss of information!
Computers:

Principles of Quantum Mechanics

• Loss of information occurs:
  - in the *measurement* procedure
  - when the system interacts with the outside world (*dissipation*)

• Computing is much faster: the loss of information is postponed to the last operation
Computers:

Principles of Quantum Mechanics

- *Measurement* implies a loss of information (Heisenberg inequalities) requires *mixed states*
- Mixed states are described by *density matrices* with evolution
  \[
  \frac{d\rho}{dt} = -i [H, \rho]
  \]
Computers:

Principles of Quantum Mechanics

- **Measurement** produces loss of information described by a *completely positive map* of the form

  \[ \mathcal{E}(\rho) = \sum E_k \rho E_k^* \]

  preserving the trace if

  \[ \sum E_k^* E_k = I. \]

- Each \( k \) represents one possible *outcome* of the measurement.
Computers:

Principles of Quantum Mechanics

- If the outcome of the measurement is given by \( k \) then the new state of the system after the measurement is given by

\[
\rho_k = \frac{E_k \rho E_k^*}{\text{Tr}(E_k \rho E_k^*)}
\]
Computers:

Principles of Quantum Mechanics

- In quantum computers, the result of a calculation is obtained through the measurement of the label indexing the digital basis.
- The algorithm has to be such that the desired result is right whatever the outcome of the measurement!!
Computers:

Principles of Quantum Mechanics

• In quantum computers, dissipative processes (interaction within or with the outside) may destroy partly the information unwillingly.
• Error-correcting codes and speed of calculation should be used to make dissipation harmless.
TO CONCLUDE (PART I):

quantum computers may work
To conclude (part I)

• The elementary unit of quantum information is the qubit, with states represented by the Bloch ball.
• Several qubits are given by tensor products leading to entanglement.
• Quantum gates are given by unitary operators and lead to quantum circuits.
• Law of physics must be considered for a quantum computer to work: measurement, dissipation...
See you next week!!