RIEMANNIAN GEOMETRY
on
METRIC CANTOR SETS

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Content

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I - Michon’s Trees

I.1)- Cantor sets
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The triadic Cantor set
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Hence without extra structure there is only one Cantor set.
I.2) - Metrics

**Definition** Let $X$ be a set. A metric $d$ on $X$ is a map $d : X \times X \mapsto \mathbb{R}_+$ such that, for all $x, y, z \in X$

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq d(x, z) + d(z, y)$. 
I.2)- Metrics

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**Definition** A metric $d$ on a set $X$ is an ultrametric if it satisfies

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

for all family $x, y, z$ of points of $C$. 
Given \((C,d)\) a metric space, for \(\epsilon > 0\) let \(\sim\) be the equivalence relation defined by

\[
x \sim y \iff \exists x_0 = x, x_1, \ldots, x_{n-1}, x_n = y \quad d(x_{k-1}, x_k) < \epsilon
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**Theorem** Let \((C, d)\) be a metric Cantor set. Then there is a sequence \(\epsilon_1 > \epsilon_2 > \cdots \epsilon_n > \cdots \geq 0\) converging to 0, such that \(\sim = \sim_n\) whenever \(\epsilon_n \geq \epsilon > \epsilon_{n+1}\).
Given \((C,d)\) a metric space, for \(\epsilon > 0\) let \(\sim^\epsilon\) be the equivalence relation defined by

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For each \(\epsilon > 0\) there is a finite number of equivalence classes and each of them is close and open.

Moreover, the sequence \([x]_{\epsilon_n}\) of clopen sets converges to \(\{x\}\) as \(n \to \infty\).
I.3) Michon’s graph
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- $\mathcal{E} = \{(v, v') \in \mathcal{V} \times \mathcal{V}; \exists n \in \mathbb{N}, v \in \mathcal{V}_n, v' \in \mathcal{V}_{n+1}, v' \subset v\}$,
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- $\delta(v) = \text{diam}\{v\}$.

The family $\mathcal{T} = (C, V, E, \delta)$ defines a weighted rooted tree, with root $C$, set of vertices $V$, set of edges $E$ and weight $\delta$. 
The Michon tree for the triadic Cantor set

\[ C = \text{root} \]

\[ \varepsilon_1 = \frac{1}{3} \]
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\[ \varepsilon_4 = \frac{1}{3^4} \]
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\[ \varepsilon_n = \frac{1}{3^n} \]
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Then $\partial T$ is the set of infinite paths starting from the root. If $v \in V$ then $[v]$ will denote the set of such paths passing through $v$. 
I.4) The boundary of a tree

Let $\mathcal{T} = (0, \forall, \mathcal{E})$ be a rooted tree. It will be called Cantorian if

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Then $\partial \mathcal{T}$ is the set of infinite path starting form the root. If $\nu \in \forall$ then $[\nu]$ will denote the set of such paths passing through $\nu$

**Theorem** The family $\{[\nu]; \nu \in \forall\}$ is the basis of a topology making $\partial \mathcal{T}$ a Cantor set.
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**Theorem** If $\mathcal{T}$ is a Cantorian rooted tree with a weight $\delta$, then $\partial \mathcal{T}$ admits a canonical ultrametric $d_\delta$ defined by

$$d_\delta(x, y) = \delta([x \land y])$$

where $[x \land y]$ is the least common ancestor of $x$ and $y$. 
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Conversely, if $\mathcal{T}$ is the Michon tree of a metric Cantor set $(C, d)$, with weight $\delta(v) = \text{diam}(v)$, then there is a contracting homeomorphism from $(C, d)$ onto $(\partial \mathcal{T}, d_\delta)$ and $d_\delta$ is the smallest ultrametric dominating $d$. 
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In particular, if $d$ is an ultrametric, then $d = d_\delta$ and the homeomorphism is an isometry.
**Theorem** Let $T$ be a Cantorian rooted tree with weight $\delta$. Then if $v \in V$, $\delta(v)$ coincides with the diameter of $[v]$ for the canonical metric.

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In particular, if $d$ is an ultrametric, then $d = d_\delta$ and the homeomorphism is an isometry.

This gives a representation of all ultrametric Cantor sets together with a parametrization of the space of ultrametrics.
I.5)- Sub-trees

A similar construction might be done by replacing the vertices by a sequence \((\Pi_n)_{n\in\mathbb{N}}\) of \textit{finite clopen partitions} such that

- \(\Pi_0\) is reduced to \(\{C\}\)
- \(\Pi_{n+1}\) is a refinement of \(\Pi_n\)
- if \(\delta_n\) is the diameter of the largest atom of \(\Pi_n\), then \(\lim_{n\to\infty} \delta_n = 0\)
- An \textit{edge} is a pair \((v, w) \in \Pi_n \times \Pi_{n+1}\), for some \(n \geq 0\) such that \(w \subset v\)

Such a tree will be \textit{reduced} if each vertex has more than one child.
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- **\(D\)** is a self-adjoint operator on \(\mathcal{H}\) with *compact resolvent* such that \([D, \pi(f)] \in \mathcal{B}(\mathcal{H})\) is a bounded operator for all \(f \in \mathcal{A}\).
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- $\mathcal{H} = \ell^2(\mathcal{V}) \otimes \mathbb{C}^2$: any $\psi \in \mathcal{H}$ will be seen as a sequence $(\psi_v)_{v \in \mathcal{V}}$ with $\psi_v \in \mathbb{C}^2$
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- $G, D$ are defined by

\[
(D\psi)_v = \frac{1}{\delta(v)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \psi_v \quad \quad (G\psi)_v = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \psi_v
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- $\mathcal{A} = \mathcal{C}_{\text{Lip}}(\mathcal{C})$ is the space of Lipshitz continuous functions on $(\mathcal{C}, d)$
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- $x, y \in [v]$
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Let $\text{Ch}(v)$ be the set of children of $v$. Consequently, the set $\Upsilon(\mathcal{C})$ of choices is given by

$$\Upsilon(\mathcal{C}) = \prod_{v \in \mathcal{V}} \Upsilon_v$$

$$\Upsilon_v = \bigsqcup_{w \neq w' \in \text{Ch}(v)} [w] \times [w']$$
The set $\mathcal{V}$ of vertices can be seen as a coarse-grained approximation of the Cantor set $C$. 
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Similarly, the set $\mathcal{Y}_v$ can be seen as a coarse-grained approximation the unit tangent vectors at $v$.

Within this interpretation, the set $\mathcal{Y}(C)$ can be seen as the unit sphere bundle inside the tangent bundle.
II.4) - Representations of $\mathcal{A}$
Let \( \tau \in \mathcal{Y}(C) \) be a choice. If \( v \in \mathcal{V} \) write \( \tau(v) = (\tau_+(v), \tau_-(v)) \). Then \( \pi_\tau \) is the representation of \( C_{\text{Lip}}(C) \) into \( \mathcal{H} \) defined by
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Let $\tau \in \mathcal{Y}(C)$ be a choice. If $v \in \mathcal{V}$ write $\tau(v) = (\tau_+(v), \tau_-(v))$. Then $\pi_\tau$ is the representation of $C_{\text{Lip}}(C)$ into $\mathcal{H}$ defined by

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II.4)- Representations of $\mathcal{A}$

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**Theorem** The distance $d$ on $C$ can be recovered from the following Connes formula

$$d(x, y) = \sup \left\{ \left| f(x) - f(y) \right| ; \sup_{\tau \in \Upsilon(C)} \| [D, \pi_\tau(f)] \| \leq 1 \right\}$$
Remark: the commutator $[D, \pi_\tau(f)]$ is given by

$$([D, \pi_\tau(f)]\psi)_v = \frac{f(\tau_+(v)) - f(\tau_-(v))}{d_\delta(\tau_+(v), \tau_-(v))} \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \psi_v$$
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In particular $\sup_\tau \|[D, \pi_\tau(f)]\|$ is the Lipschitz norm of $f$

$$\|f\|_{\text{Lip}} = \sup_{x \neq y \in \mathbb{C}} \left| \frac{f(x) - f(y)}{d_\delta(x, y)} \right|$$
III - \(\zeta\)-function and Metric Measure


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The $\zeta$-function of the Dirac operator is defined by

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The ζ-function of the Dirac operator is defined by

\[ \zeta(s) = \text{Tr} \left( \frac{1}{|D|^s} \right), \quad s \in \mathbb{C} \]

The abscissa of convergence is the smallest positive real number \( s_0 > 0 \) so that the series defined by the trace above converges for \( \Re(s) > s_0 \).
III.1) - \( \zeta \)-function

The \( \zeta \)-function of the Dirac operator is defined by

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Thanks to the definition of the Dirac operator

\[
\zeta(s) = 2 \sum_{v \in \mathcal{V}} \delta(v)^s
\]
Theorem Let \((C,d)\) be an ultrametric Cantor set associated with a reduced Michon tree.

- The abscissa of convergence of the \(\zeta\)-function of the corresponding Dirac operator is always larger than or equal to the Hausdorff dimension of \((C,d)\).

- If the Hausdorff dimension is finite, then there is a choice of the Michon tree so that \(s_0 = \dim_H(C,d)\).
III.2) Dixmier Trace & Metric Measure
If the abscissa of convergence is finite, then a \textit{probability measure} \( \mu \) on \((C, d)\) can be defined as follows (if the limit exists)

\[
\mu(f) = \lim_{s \downarrow s_0} \frac{\text{Tr} \left( |D|^{-s} \pi_\tau(f) \right)}{\text{Tr} \left( |D|^{-s} \right)} \\
\text{for } f \in C_{\text{Lip}}(C)
\]
If the abscissa of convergence is finite, then a *probability measure* \( \mu \) on \((C,d)\) can be defined as follows (if the limit exists)

\[
\mu(f) = \lim_{s \downarrow s_0} \frac{\text{Tr} (|D|^{-s}\pi_\tau(f))}{\text{Tr} (|D|^{-s})} \quad f \in \mathcal{C}_{\text{Lip}}(C)
\]

This limit coincides with the *normalized Dixmier trace*

\[
\frac{\text{Tr}_{\text{Dix}} (|D|^{-s_0}\pi_\tau(f))}{\text{Tr}_{\text{Dix}} (|D|^{-s_0})}
\]
Theorem

- The definition of the measure $\mu$ is independent of the choice $\tau$.
- The Dixmier trace is unique if and only if the Hausdorff measure of $(C,d)$ exists, is positive and finite.
- In the latter case $\mu$ coincides with the normalized Hausdorff measure of $(C,d)$. 
If $\zeta$ admits an *isolated simple pole at* $s = s_0$, then $|D|^{-1}$ belongs to the *Mačaev ideal* $\mathcal{L}^{s_0+}(\mathcal{H})$. Therefore the measure $\mu$ is well defined.
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• In particular $\mu$ is the metric analog of the Lebesgue measure class on a Riemannian manifold, in that the measure of a ball of radius $r$ behaves like $r^{s_0}$ for $r$ small

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\mu(B(x, r)) \downarrow 0 \sim r^{s_0}
\]

• $\mu$ is the analog of the volume form on a Riemannian manifold.
As a consequence $\mu$ defines a canonical probability measure $\nu$ on the space of choices $\Upsilon$ as follows

$$\nu = \bigotimes_{v \in \mathcal{V}} \nu_v$$

$$\nu_v = \frac{1}{Z_v} \sum_{w \neq w' \in \text{Ch}(v)} \mu \otimes \mu |_{[w] \times [w]}$$

where $Z_v$ is a normalization constant given by

$$Z_v = \sum_{w \neq w' \in \text{Ch}(v)} \mu([w]) \mu([w'])$$
IV - The Laplace-Beltrami Operator


J. Pearson, J. Bellissard,
*Noncommutative Riemannian Geometry and Diffusion on Ultrametric Cantor Sets*,

A. Julien, J. Savinien,
IV.1)- Dirichlet Forms
IV.1) Dirichlet Forms

Let \((X, \mu)\) be a probability space. For \(f\) a real valued measurable function on \(X\), let \(\hat{f}\) be the function obtained as

\[
\hat{f}(x) = \begin{cases} 
1 & \text{if } f(x) \geq 1 \\
 f(x) & \text{if } 0 \leq f(x) \leq 1 \\
0 & \text{if } f(x) \leq 0
\end{cases}
\]
Markovian cut-off of a real valued function
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- \(Q\) is densely defined with domain \(\mathcal{D} \subset L^2(X, \mu)\)
- \(Q\) is closed
- \(Q\) is Markovian, namely if \(f \in \mathcal{D}\), then \(Q(\hat{f}, \hat{f}) \leq Q(f, f)\)
The simplest typical example of Dirichlet form is related to the Laplacian $\Delta_\Omega$ on a bounded domain $\Omega \subset \mathbb{R}^D$

$$Q_\Omega(f, g) = \int_\Omega d^p x \, \nabla f(x) \cdot \nabla g(x)$$

with domain $\mathcal{D} = C^1_0(\Omega)$ the space of continuously differentiable functions on $\Omega$ vanishing on the boundary.

*This form is closeable in $L^2(\Omega)$ and its closure defines a Dirichlet form.*
Any closed positive sesquilinear form $Q$ on a Hilbert space, defines canonically a *positive self-adjoint operator* $-\Delta_Q$ satisfying

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If $Q$ is a Dirichlet form on $X$, then the contraction semigroup $\Phi = (\Phi_t)_{t \geq 0}$ is a Markov semigroup.
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**Theorem (Fukushima)** A contraction semi-group on $L^2(X, \mu)$ is a Markov semi-group if and only if its generator is defined by a Dirichlet form.
IV.2) The Laplace-Beltrami Form
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Let $M$ be a *Riemannian manifold* of dimension $D$. The *Laplace-Beltrami operator* is associated with the Dirichlet form
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$$Q_M(f, g) = \sum_{i,j=1}^{D} \int_M d^Dx \sqrt{\det(g(x))} g^{ij}(x) \partial_i f(x) \partial_j g(x)$$

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IV.2)- The Laplace-Beltrami Form

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where $g$ is the metric. Equivalently (in local coordinates)

$$Q_M(f, g) = \int_M d^p x \sqrt{\det(g(x))} \ \int_{S(x)} d\nu_x(u) \ u \cdot \nabla f(x) \ u \cdot \nabla g(x)$$
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where $S(x)$ represents the *unit sphere* in the tangent space whereas $\nu_x$ is the *normalized Haar measure* on $S(x)$. 
Similarly, if \((C, d)\) is an ultrametric Cantor set, the expression

\[
[D, \pi_\tau(f)]
\]

can be interpreted as a *directional derivative*, analogous to \(u \cdot \nabla f\), since a choice \(\tau\) has been interpreted as a unit tangent vector.
Similarly, if \((C,d)\) is an ultrametric Cantor set, the expression

\[
[D, \pi_\tau(f)]
\]

can be interpreted as a *directional derivative*, analogous to \(u \cdot \nabla f\), since a choice \(\tau\) has been interpreted as a unit tangent vector.

The *Laplace-Pearson operators* are defined, by analogy, by

\[
Q_s(f,g) = \int_\gamma d\nu(\tau) \operatorname{Tr} \left\{ \frac{1}{|D|^s} [D, \pi_\tau(f)]^* [D, \pi_\tau(g)] \right\}
\]

for \(f, g \in C_{\text{Lip}}(C)\) and \(s > 0\).
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**Theorem** For any $s \in \mathbb{R}$, the form $Q_s$ defined on $\mathcal{D}$ is closeable on $L^2(C, \mu)$ and its closure is a Dirichlet form.
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The corresponding operator $-\Delta_s$ leaves $\mathcal{D}$ invariant, has a discrete spectrum.
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For $s < s_0 + 2$, $-\Delta_s$ is unbounded with compact resolvent.
IV.3)- Jumps Process over Gaps
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$\Delta_s$ generates a Markov semigroup, thus a stochastic process \((X_t)_{t \geq 0}\) where the \(X_t\)'s takes on values in \(C\).
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Given $v \in \mathcal{V}$, its spine is the set of vertices located along the finite path joining the root to $v$. The vine $\mathcal{V}(v)$ of $v$ is the set of vertices $w$, not in the spine, which are children of one vertex of the spine.
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Then if $\chi_v$ is the characteristic function of $[v]$

$$\Delta_s \chi_v = \sum_{w \in \mathcal{V}(v)} p(v, w)(\chi_w - \chi_v)$$
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\[
\Delta_s \chi_v = \sum_{w \in \mathcal{V}(v)} p(v, w)(\chi_w - \chi_v)
\]

where \(p(v, w) > 0\) represents the probability for \(X_t\) to jump from \(v\) to \(w\) per unit time.
The vine of a vertex $v$

$v = v_n$

$[w_n] \subseteq [v_{n-1}] \setminus [v_n]$
Jump process from $v$ to $w$
The tree for the triadic ring $\mathbb{Z}(3)$
Jump process in $\mathbb{Z}(3)$

$\text{Prob}_{\text{jump}} \sim 3^{-1(4-s)}$
$\text{Prob}(\text{jump}) \sim 3^{-2(4-s)}$

Jump process in $\mathbb{Z}(3)$
\[ \text{Prob}_{\text{jump}} \sim 3^{-3(4-s)} \]

Jump process in $\mathbb{Z}(3)$
Concretely, if $\hat{w}$ denotes the father of $w$ (which belongs to the spine)

$$p(v, w) = 2\delta(\hat{w})^{s-2} \frac{\mu([v])}{Z_{\hat{w}}}$$

where $Z_{\hat{w}}$ is the normalization constant for the measure $\nu_{\hat{w}}$ on the set of choices at $\hat{w}$, namely

$$Z_{\hat{w}} = \sum_{u \neq u' \in \text{Ch}(\hat{w})} \mu([u])\mu([u'])$$
IV.4)- Eigenspaces

Let $v$ be a vertex of the Michon graph with $\text{Ch}(v)$ as its set of children.
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Let \( v \) be a vertex of the Michon graph with \( \text{Ch}(v) \) as its set of children. Let \( \mathcal{E}_v \) be the linear space generated by the characteristic function \( \chi_w \) of the \( [w]'s \) with \( w \in \text{Ch}(v) \).
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$$\chi_v = \sum_{w \in \text{Ch}(v)} \chi_w \in \mathcal{E}_v.$$ 

**Theorem** For any $s \in \mathbb{R}$, the eigenspaces of $-\Delta_s$ are the spaces of the form $\{\chi_v\}^\perp \subset \mathcal{E}_v$, namely, the orthogonal complement of $\chi_v$ is $\mathcal{E}_v$. 

V - To conclude
• Ultrametric Cantor sets can be described as *Riemannian manifolds*, through Noncommutative Geometry.

• An analog of the *tangent unit sphere* is given by *choices*

• The *Hausdorff dimension* plays the role of the dimension

• A *volume measure* is defined through the Dixmier trace

• A *Laplace-Beltrami operator* is defined with compact resolvent and Weyl asymptotics

• It generates a *jump process* playing the role of the *Brownian motion*.

• This process exhibits *anomalous diffusion*. 
Recent Progress


• The construction of a spectral triple can be extended to any **compact metric space** if the partitions by clopen sets are replaces by suitable **open covers**.

• If the compact metric space \((X, d)\) has **finite Hausdorff dimension** then the spectral triple can be chosen to admits \(\dim_H(X)\) as **abscissa of convergence**.

• If \((X, d)\) admits a **positive finite Hausdorff measure** the spectral triple can be constructed so as to have the measure \(\mu\), defined by the Dixmier trace, equal to the **normalized Hausdorff measure**.

• Under some extra local regularity property on \((X, d)\) a Laplace-Beltrami operator be defined (J. Cheeger).