Periodic Approximants to Aperiodic Hamiltonians

Jean BELLISSARD
Westfälische Wilhelms-Universität, Münster
Department of Mathematics

Georgia Institute of Technology, Atlanta
School of Mathematics & School of Physics
e-mail: jeanbel@math.gatech.edu

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CRC 701, Bielefeld, Germany
SFB 878, Münster, Germany

Groups, Geometry & Actions
Contributors

G. De Nittis, *Department Mathematik*, Friedrich-Alexander Universität, Erlangen-Nürnberg, Germany

S. Beckus, *Mathematisches Institut*, Friedrich-Schiller-Universität Jena, Germany

V. Milani, *Dep. of Mathematics*, Shahid Beheshti University Tehran, Iran
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online at  http://people.math.gatech.edu/~jeanbel/talksjbE.html
Content

Warning This talk is reporting on a work in progress.

1. Motivation
2. Continuous Fields
3. Conclusion
I - Motivations
Motivation

Spectrum of the Kohmoto model
(Fibonacci Hamiltonian)

\[(H\psi)(n) = \psi(n+1) + \psi(n-1) + \lambda \chi_{(0,\alpha)}(x-n\alpha) \psi(n)\]
as a function of \(\alpha\).

Method:
transfer matrix calculation


Renormalization of Quasiperiodic Mappings
Stellan Otlund and Seung-hwan Kim
Motivation

Fig. 3. – We show, respectively, the IDOS of the Octonacci chain (up) and the IDOS of the labyrinth, for a) $\tau = 0.8$ (no gap, finite measure), b) $\tau = 0.6$ (some gaps and finite measure) and c) $\tau = 0.3$ (infinity of gaps and zero measure). The energy varies between $-2$ and $2$, since $\tau < 1$.

C. Sire

Electronic Spectrum of a 2D Quasi-Crystal Related to the Octagonal Quasi-Periodic Tiling.
EUROPHYSICS LETTERS

Solvable 2D-model, reducible to 1D-calculations
Motivation

A sample of the icosahedral quasicrystal AlPdMn
Methodologies

• For one dimensional Schrödinger equation of the form

\[ H\psi(x) = -\frac{d^2\psi}{dx^2} + V(x)\psi(x) \]

a transfer matrix approach has been used for a long time to analyze the spectral properties (Bogoliubov ’36).

• A KAM-type perturbation theory has been used successfully (Dinaburg, Sinai ’76, JB ’80’s).
Methodologies

• For discrete one-dimensional models of the form

\[ H\psi(n) = t_{n+1}\psi(n+1) + t_n\psi(n-1) + V_n\psi(n) \]

a transfer matrix approach is the most efficient method, both for numerical calculation and for mathematical approach:

– the KAM-type perturbation theory also applies (JB ’80’s).
– models defined by substitutions using the trace map
  (Khomoto et al., Ostlundt et al. ‘83, JB ‘89, JB, Bovier, Ghez, Damanik... in the nineties)
– theory of cocycle (Avila, Jitomirskaya, Damanik, Krikorian, Gorodetsky...).
Methodologies

- In higher dimension almost no rigorous results are available
- Exceptions are for models that are Cartesian products of 1D models (Sire ‘89, Damanik, Gorodestky, Solomyak ‘14)
- Numerical calculations performed on quasi-crystals have shown that
  - Finite cluster calculation lead to a large number of *spurious edge states*.
  - *Periodic approximations* are much more efficient
  - Some periodic approximations exhibit *defects* giving *contributions* in the energy spectrum.
II - Continuous Fields
Continuous Fields of Hamiltonians

\[ A = (A_t)_{t \in T} \text{ is a field of self-adjoint operators whenever} \]

1. \( T \) is a topological space,
2. for each \( t \in T, \mathcal{H}_t \) is a Hilbert space,
3. for each \( t \in T, A_t \) is a self-adjoint operator acting on \( \mathcal{H}_t \).

The field \( A = (A_t)_{t \in T} \) is called \textit{p2-continuous} whenever, for every polynomial \( p \in \mathbb{R}(X) \) with degree at most 2, the following norm map is \textit{continuous}

\[ \Phi_p : t \in T \mapsto \|p(A_t)\| \in [0, +\infty) \]
Continuous Fields of Hamiltonians

**Theorem:** (S. Beckus, J. Bellissard ‘16)

1. A field \( A = (A_t)_{t \in T} \) of self-adjoint bounded operators is \( p2 \)-continuous if and only if the spectrum of \( A_t \), seen as a compact subset of \( \mathbb{R} \), is a continuous function of \( t \) with respect to the Hausdorff metric.

2. Equivalently \( A = (A_t)_{t \in T} \) is \( p2 \)-continuous if and only if the spectral gap edges of \( A_t \) are continuous functions of \( t \).
Continuous Fields of Hamiltonians

The field $A = (A_t)_{t \in T}$ is called $p2$-$\alpha$-Hölder continuous whenever, for every polynomial $p \in \mathbb{R}(X)$ with degree at most 2, the following norm map is $\alpha$-Hölder continuous

$$\Phi_p : t \in T \mapsto \|p(A_t)\| \in [0, +\infty)$$

uniformly w.r.t. $p(X) = p_0 + p_1 X + p_2 X^2 \in \mathbb{R}(X)$ such that $|p_0| + |p_1| + |p_2| \leq M$, for some $M > 0$. 
Continuous Fields of Hamiltonians

**Theorem:** (S. Beckus, J. Bellissard ‘16)

1. A field $A = (A_t)_{t \in T}$ of self-adjoint bounded operators is $p^2\alpha$-Hölder continuous then the spectrum of $A_t$, seen as a compact subset of $\mathbb{R}$, is an $\alpha/2$-Hölder continuous function of $t$ with respect to the Hausdorff metric.

2. In such a case, the edges of a spectral gap of $A_t$ are $\alpha$-Hölder continuous functions of $t$ at each point $t$ where the gap is open.

3. At any point $t_0$ for which a spectral gap of $A_t$ is closing, if the tip of the gap is isolated from other gaps, then its edges are $\alpha/2$-Hölder continuous functions of $t$ at $t_0$.

4. Conversely if the gap edges are $\alpha$-Hölder continuous, then the field $A$ is $p^2\alpha$-Hölder continuous.
The spectrum of the Harper model
the Hamiltonina is $p^2$-Lipschitz continuous

(JB, ’94)
Continuous Fields on C*-algebras

(Tomyama 1958, Dixmier-Douady 1962)

Given a topological space $T$, let $\mathcal{A} = (\mathcal{A}_t)_{t \in T}$ be a family of C*-algebras. A vector field is a family $a = (a_t)_{t \in T}$ with $a_t \in \mathcal{A}_t$ for all $t \in T$. $\mathcal{A}$ is called continuous whenever there is a family $\gamma$ of vector fields such that,

- for all $t \in T$, the set $\gamma_t$ of elements $a_t$ with $a \in \gamma$ is a dense *-subalgebra of $\mathcal{A}_t$
- for all $a \in \gamma$ the map $t \in T \mapsto \|a_t\| \in [0, +\infty)$ is continuous
- a vector field $b = (b_t)_{t \in T}$ belongs to $\gamma$ if and only if, for any $t_0 \in T$ and any $\epsilon > 0$, there is $U$ an open neighborhood of $t_0$ and $a \in \gamma$, with $\|a_t - b_t\| < \epsilon$ whenever $t \in U$. 
Theorem If $\mathcal{A}$ is a continuous field of $C^*$-algebras and if $a \in \gamma$ is a continuous self-adjoint vector field, then, for any continuous function $f \in C_0(\mathbb{R})$, the maps $t \in T \mapsto \|f(a_t)\| \in [0, +\infty)$ are continuous.

In particular, such a vector field is $p2$-continuous.
A groupoid $G$ is a category the object of which $G_0$ and the morphism of which $G$ make up two sets. More precisely

- there are two maps $r, s : G \to G_0$ (range and source)
- $(\gamma, \gamma') \in G_2$ are compatible whenever $s(\gamma) = r(\gamma')$
- there is an associative composition law $(\gamma, \gamma') \in G_2 \mapsto \gamma \circ \gamma' \in G$, such that $r(\gamma \circ \gamma') = r(\gamma)$ and $s(\gamma \circ \gamma') = s(\gamma')$
- a unit $e$ is an element of $G$ such that $e \circ \gamma = \gamma$ and $\gamma' \circ e = \gamma'$ whenever compatibility holds; then $r(e) = s(e)$ and the map $e \to x = r(e) = s(e) \in G_0$ is a bijection between units and objects;
- each $\gamma \in G$ admits an inverse such that $\gamma \circ \gamma^{-1} = r(\gamma) = s(\gamma^{-1})$ and $\gamma^{-1} \circ \gamma = s(\gamma) = r(\gamma^{-1})$
Locally Compact Groupoids

• A groupoid $G$ is *locally compact* whenever
  – $G$ is endowed with a locally compact Hausdorff topology
  – the maps $r, s$, the *composition* and the *inverse* are *continuous* functions.

  Then the set of units is a closed subset of $G$.

• A *Haar system* is a family $\lambda = (\lambda^x)_{x \in G_0}$ of positive Borel measures on the fibers $G^x = r^{-1}(x)$, such that
  – if $\gamma : x \to y$, then $\gamma^* \lambda^x = \lambda^y$
  – if $f \in C_c(G)$ is continuous with compact support, then the map $x \in G_0 \mapsto \lambda^x(f)$ is *continuous*. 
Locally Compact Groupoids

Example:

Let $\Omega$ be a compact Hausdorff space, let $G$ be a locally compact group acting on $\Omega$ by homeomorphisms. Then $\Gamma = \Omega \times G$ becomes a locally compact groupoid as follows

- $\Gamma_0 = \Omega$, is the set of units,
- $r(\omega, g) = \omega$ and $s(\omega, g) = g^{-1}\omega$
- $(\omega, g) \circ (g^{-1}\omega, h) = (\omega, gh)$
- Each fiber $\Gamma^\omega \simeq G$, so that if $\mu$ is the Haar measure on $G$, it gives a Haar system $\lambda$ with $\lambda^\omega = \mu$ for all $\omega \in \Omega$.

This groupoid is called the crossed-product and is denoted $\Omega \rtimes G$
Groupoid $C^*$-algebra

Let $G$ be a locally compact groupoid with a Haar system $\lambda$. Then the complex vector space $C_c(G)$ of complex valued continuous functions with compact support on $G$ becomes a $*$-algebra as follows:

- **Product (convolution):**

  $$ab(\gamma) = \int_{G_x} a(\gamma') b(\gamma'^{-1} \circ \gamma) \, d\lambda^x(\gamma') \quad x = r(\gamma)$$

- **Adjoint:**

  $$a^*(\gamma) = a(\gamma^{-1})$$
Groupoid C*-algebra

The following construction gives a C*-norm

• for each $x \in G_0$, let $\mathcal{H}_x = L^2(G^x, \lambda^x)$
• for $a \in C_c(G)$, let $\pi_x(a)$ be the operator on $\mathcal{H}_x$ defined by

$$\pi_x(a)\psi(\gamma) = \int_{G^x} a(\gamma^{-1} \circ \gamma') \psi(\gamma')d\lambda^x(\gamma')$$

• $(\pi_x)_{x \in G_0}$ gives a faithful covariant family of *-representations of $C_c(G)$, namely if $\gamma : x \rightarrow y$ then $\pi_x \sim \pi_y$.

• then $\|a\| = \sup_{x \in G_0} \|\pi_x(a)\|$ is a C*-norm; the completion of $C_c(G)$ with respect to this norm is called the reduced C*-algebra of $G$ and is denoted by $C^*_{red}(G)$. 
Continuous Fields of Groupoids

(N. P. Landsman, B. Ramazan, 2001)

- A **field of groupoid** is a triple \((G, T, p)\), where \(G\) is a groupoid, \(T\) a set and \(p : G \to T\) a map, such that, if \(p_0 = p \mathrel{|}_{G_0}\), then \(p = p_0 \circ r = p_0 \circ s\).

- Then the subset \(G_t = p^{-1}\{t\}\) is a groupoid depending on \(t\).

- If \(G\) is **locally compact**, \(T\) a **Hausdorff** topological space and \(p\) **continuous** and **open**, then \((G, T, P) = (G_t)_{t \in T}\) is called a **continuous field of groupoids**.
Theorem: (N. P. Landsman, B. Ramazan, 2001)

Let \((G, T, p)\) be a continuous field of locally compact groupoids with Haar systems. If \(G_t\) is amenable for all \(t \in T\), then the field \(\mathcal{A} = (\mathcal{A}_t)_{t \in T}\) of \(C^*\)-algebras defined by \(\mathcal{A}_t = C^*(G_t)\) is continuous.
III - Tautological Groupoid
Periodic Approximations

Approximating an aperiodic system by a periodic one makes sense within the following framework

- $\Omega$ is a *compact* Hausdorff metrizable space,
- a locally compact *group* $G$ acts on $\Omega$ by homeomorphisms,
- $I(\Omega)$ is the set of *closed $G$-invariant* subsets of $\Omega$:
  - a subset $M \in I(\Omega)$ is *minimal* if all its $G$-orbits are dense.
  - a point $\omega \in \Omega$ is called *periodic* if there is a *uniform lattice* $\Lambda \subset G$ such that $g\omega = \omega$ for $g \in \Lambda$. In such a case $\text{Orb}(\omega)$ is a quotient of $G/\Lambda$, and is thus is compact.
  - if $G$ is *discrete*, any periodic orbit is a *finite set*. 
Periodic Approximations

Example 1)- Subshifts

Let $\mathcal{A}$ be a finite set (alphabet). Let $\Omega = \mathcal{A}^\mathbb{Z}$: it is compact for the product topology. The shift operator $S$ defines a $\mathbb{Z}$-action.

1. A sequence $\xi = (x_n)_{n \in \mathbb{Z}}$ is periodic if and only if $\xi$ can be written as an infinite repetition of a finite word. The $S$-orbit of $\xi$ is then finite.

2. The set of periodic points of $\Omega$ is dense.

3. A subshift is provided by a closed $S$-invariant subset, namely a point in $\mathcal{I}(\Omega)$. 
Periodic Approximations

Example 2)- Delone Sets A Delone set $\mathcal{L} \subset \mathbb{R}^d$ is

- a discrete closed subset,
- there is $0 < r$ such that each ball of radius $r$ intersects $\mathcal{L}$ at one point at most,
- there is $0 < R$ such that each ball of radius $R$ intersects $\mathcal{L}$ at one point at least.

Then

1. the set $\Omega = \text{Del}_{r,R}$ of such Delone sets in $\mathbb{R}^d$ can be endowed with a topology that makes it compact,
2. the group $\mathbb{R}^d$ acts on $\Omega$ by homeomorphisms,
3. the periodic Delone sets make up a dense subset in $\Omega$. 
Periodic Approximations

**Question:** in which sense can one approximate a minimal infinite $G$-invariant subset by a sequence of periodic orbits?
The Fell and Vietoris Topologies

(Vietoris 1922, Fell 1962)

Given a topological space $X$, let $\mathcal{C}(X)$ be the set of closed subsets of $X$.

Let $F \subset X$ be closed and let $\mathcal{F}$ be a finite family of open sets. Then

$$U(F, \mathcal{F}) = \{G \in \mathcal{C}(X) ; G \cap F = \emptyset, G \cap O \neq \emptyset, \forall O \in \mathcal{F}\}$$

Then the family of $U(F, \mathcal{F})$'s is a basis for a topology called the Vietoris topology.

Replacing $F$ by a compact set $K$, the same definition leads to the Fell topology.
The Fell and Vietoris Topologies

• $C(X)$ is *Fell-compact*,
• if $X$ is locally compact and Hausdorff, $C(X)$ is *Hausdorff* for both Fell and Vietoris,
• if $(X, d)$ is a *complete metric* space, the Vietoris topology coincides with the topology defined by the *Hausdorff metric*.
• If $X$ is *compact* both topologies *coincide*.

**Theorem** If $(\Omega, G)$ is a topological dynamical system, the set $I(\Omega)$ is compact for both the Fell and the Vietoris topologies.
The Fell and Vietoris Topologies

Example:

If \((\Omega = \mathcal{A}^\mathbb{Z}, S)\), periodic orbits \textit{ARE NOT} Vietoris-dense in \(I(\Omega)\)

For instance, if \(\mathcal{A} = \{0, 1\}\) let \(\xi_0 \in \Omega\) be the sequence defined by

\[
\xi_0 = (x_n)_{n \in \mathbb{Z}} \quad \quad \quad x_n = \begin{cases} 
0 & \text{if } n < 0 \\
1 & \text{if } n \geq 0
\end{cases}
\]

Then \(\text{Orb}(\xi)\) is \textit{isolated} in \(I(\Omega)\) for the Vietoris topology.
The Tautological Groupoid

- Let \( \mathcal{I}(\Omega) \subset \mathcal{I}(\Omega) \times \Omega \) be the set of pairs \((M, \omega)\) such that \( \omega \in M \). Endowed with the product topology it is compact Hausdorff.
- \( G \) acts on it by homeomorphisms through \( g(M, \omega) = (M, g\omega) \).
- Let \( \Gamma = \mathcal{I}(\Omega) \rtimes G \), let \( T = \mathcal{I}(\Omega) \) and let \( p : \Gamma \to T \) defined by
  \[
p(M, \omega, g) = M
  \]
  
  Then \((G, T, p)\) is a continuous field of locally compact groupoids.
- If \( G \) is amenable, then the family \( \mathcal{A} = (\mathcal{A}_M)_{M \in \mathcal{I}(\Omega)} \) where \( \mathcal{A}_M = C^*(\Gamma_M) \) gives a continuous field of \( C^* \)-algebras.
The Tautological Groupoid

Hence

If $M$ is a closed invariant subset of $\Omega$ that is a Vietoris-limit point of the set of periodic orbit, then any continuous field of Hamiltonian in $\mathcal{A}$ has a spectrum that can be approximated by the spectrum of a suitable sequence of periodic approximations.

**Question:** Which invariant subsets of $\Omega$ are Vietoris limit points of periodic orbits?
IV - Periodic Approximations for Subshifts
Subshifts

Let $\mathcal{A}$ be a finite alphabet, let $\Omega = \mathcal{A}^\mathbb{Z}$ be equipped with the shift $S$. Let $\Sigma \in \mathcal{J}(\Omega)$ be a subshift. Then

- given $l, r \in \mathbb{N}$ an $(l, r)$-collared dot is a dotted word of the form $u \cdot v$ with $u, v$ being words of length $|u| = l, |v| = r$ such that $uv$ is a sub-word of at least one element of $\Sigma$
- an $(l, r)$-collared letter is a dotted word of the form $u \cdot a \cdot v$ with $a \in \mathcal{A}, u, v$ being words of length $|u| = l, |v| = r$ such that $uav$ is a sub-word of at least one element of $\Sigma$: a collared letter links two collared dots
- let $\mathcal{V}_{l,r}$ be the set of $(l, r)$-collared dots, let $\mathcal{E}_{l,r}$ be the set of $(l, r)$-collared letters: then the pair $\mathcal{G}_{l,r} = (\mathcal{V}_{l,r}, \mathcal{E}_{l,r})$ gives a finite directed graph

(de Bruijn, ‘46, Anderson-Putnam ‘98, Gähler, ‘01)
The Fibonacci Tiling

- **Alphabet:** $A = \{a, b\}$
- **Fibonacci sequence:** generated by the substitution $a \rightarrow ab$, $b \rightarrow a$ starting from either $a \cdot a$ or $b \cdot a$

Left: $G_{1,1}$

Right: $G_{8,8}$
The Thue-Morse Tiling

- **Alphabet:** $A = \{a, b\}$
- **Thue-Morse sequences:** generated by the substitution $a \rightarrow ab$, $b \rightarrow ba$ starting from either $a \cdot a$ or $b \cdot a$
The Rudin-Shapiro Tiling

- **Alphabet**: $\mathcal{A} = \{a, b, c, d\}$
- **Rudin-Shapiro sequences**: generated by the substitution $a \rightarrow ab$, $b \rightarrow ac$, $c \rightarrow db$, $d \rightarrow dc$ starting from either $b \cdot a$, $c \cdot a$ or $b \cdot d$, $c \cdot d$
The Full Shift on Two Letters

- **Alphabet**: \( \mathcal{A} = \{a, b\} \) all possible word allowed.
**Strongly Connected Graphs**

The de Bruijn graphs are

- *simple*: between two vertices there is at most one edge,
- *connected*: if the sub-shift is *topologically transitive*, (i.e. one orbit is dense), then between any two vertices, there is at least one path connected them,
- has *no dangling vertex*: each vertex admits at least one ingoing and one outgoing vertex,
- if $n = l + r = l' + r'$ then the graphs $G_{l,r}$ and $G_{l',r'}$ are *isomorphic* and denoted by $G_n$. 
A directed graph is called *strongly connected* if any pair $x, y$ of vertices there is an *oriented path* from $x$ to $y$ and another one from $y$ to $x$.

**Proposition:** If the sub-shift $\Sigma$ is minimal (i.e. every orbit is dense), then each of the de Bruijn graph is strongly connected.

**Main result:**

**Theorem:** A subshift $\Sigma \subset A^\mathbb{Z}$ can be Vietoris approximated by a sequence of periodic orbits if and only if it admits is a sequence of strongly connected de Bruijn graphs.
Question:

Is there a similar criterion for the space of Delone sets in $\mathbb{R}^d$ or for some remarkable subclasses of it?

Some sufficient conditions have been found for $\Omega = A^G$, where $G$ is a discrete, countable and amenable group, in particular when $G = \mathbb{Z}^d$.

(S. Beckus, PhD Thesis, 2016)
Thanks for listening!

Now it’s time for tea!