VARIOUS MATHEMATICAL ASPECTS of TILING SPACES

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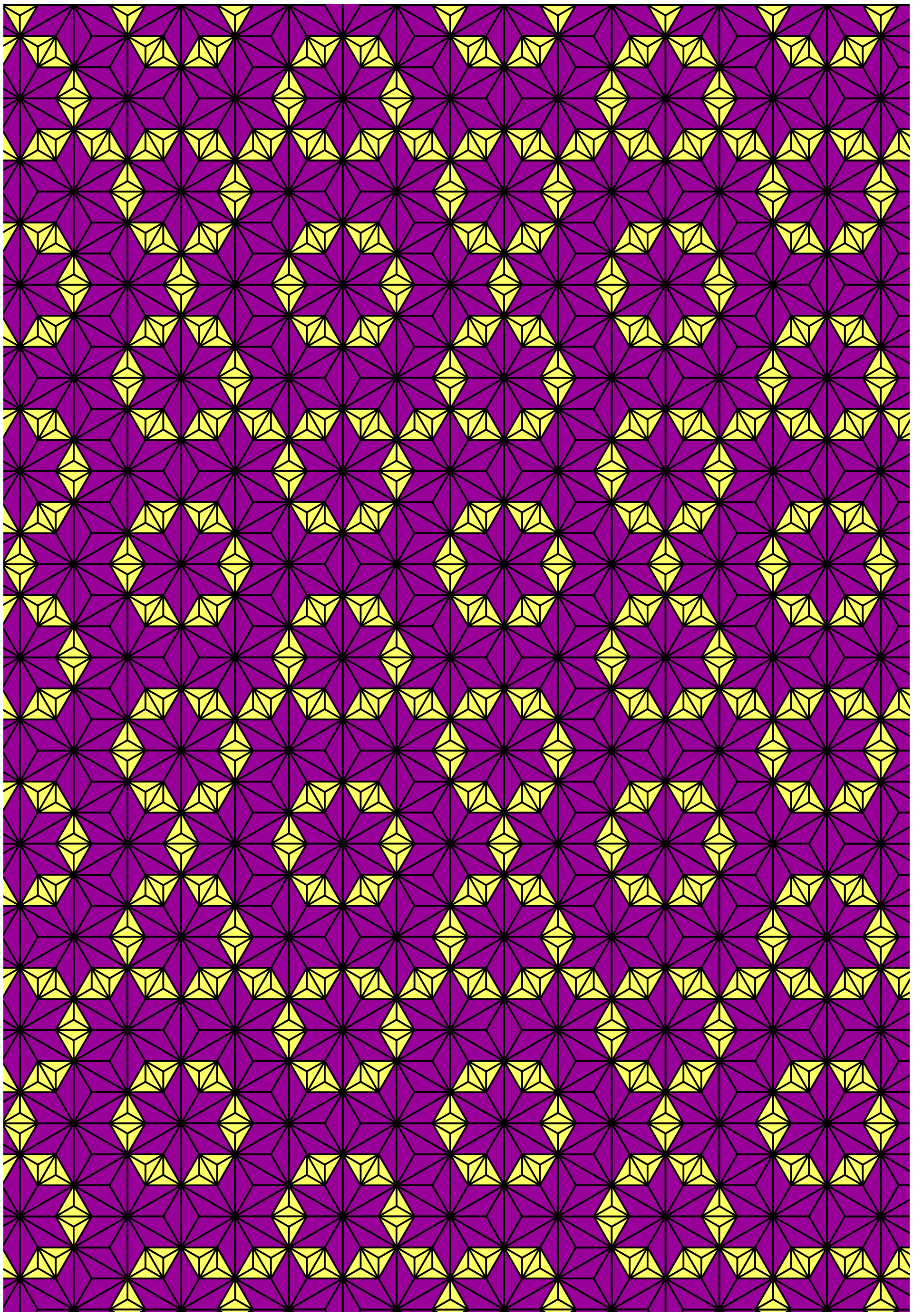
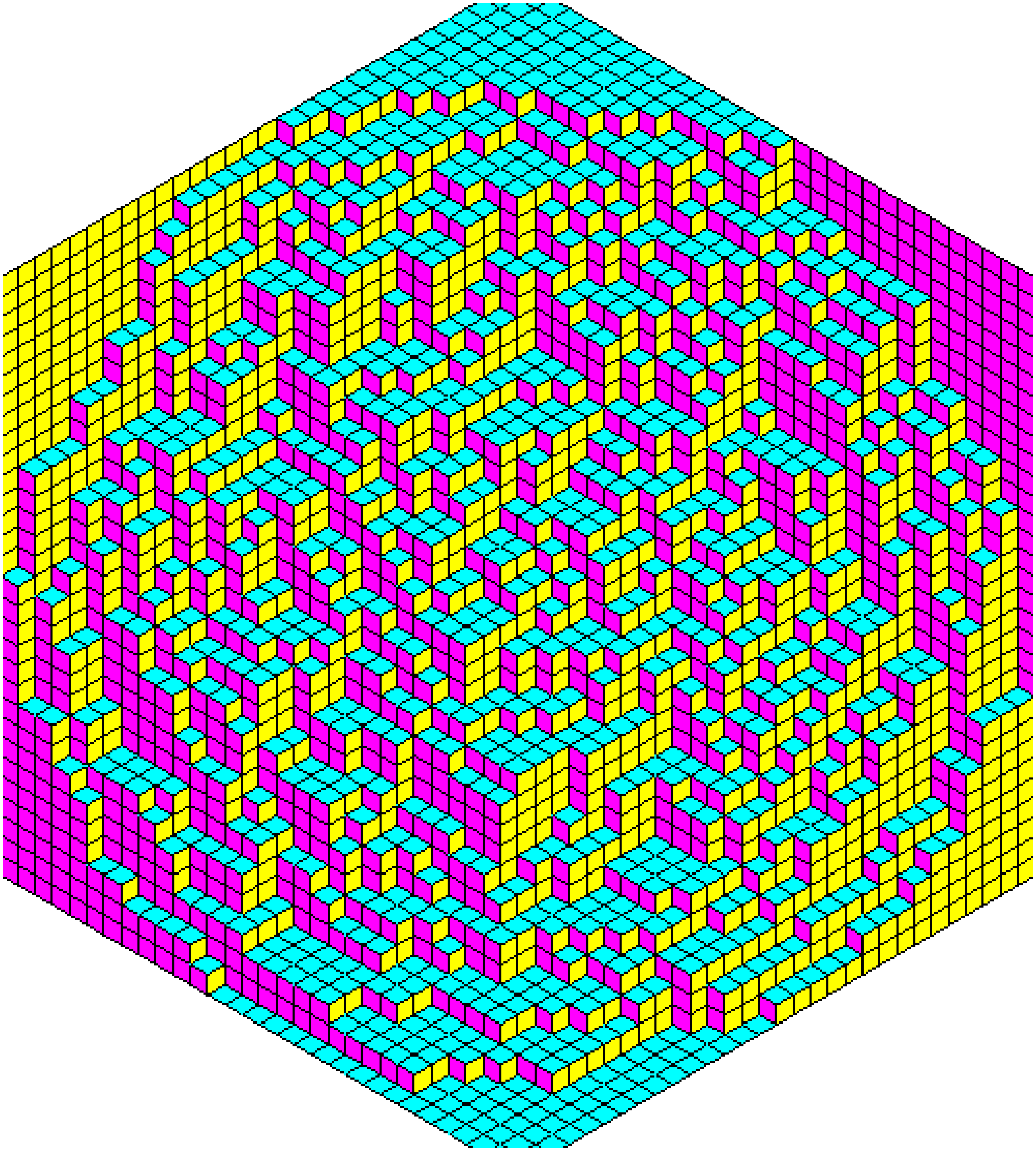
Content

I - Tilings, Tilings,...



- A triangle tiling -



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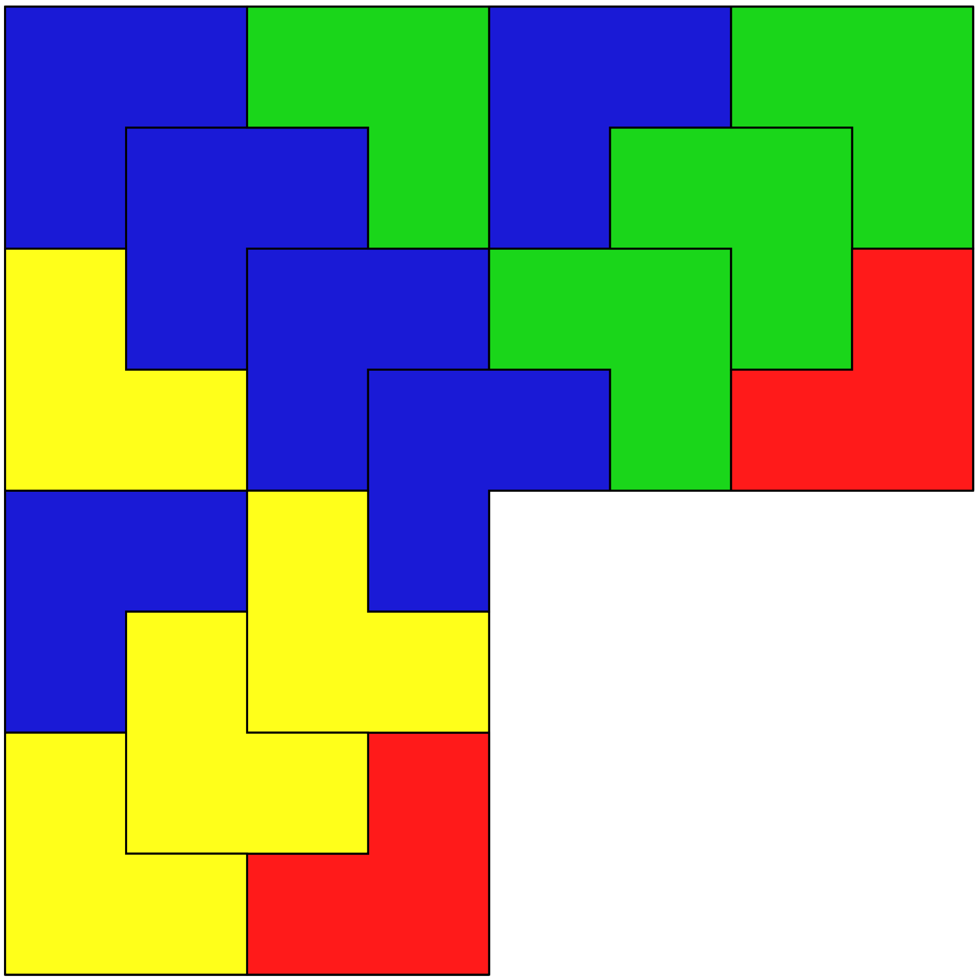

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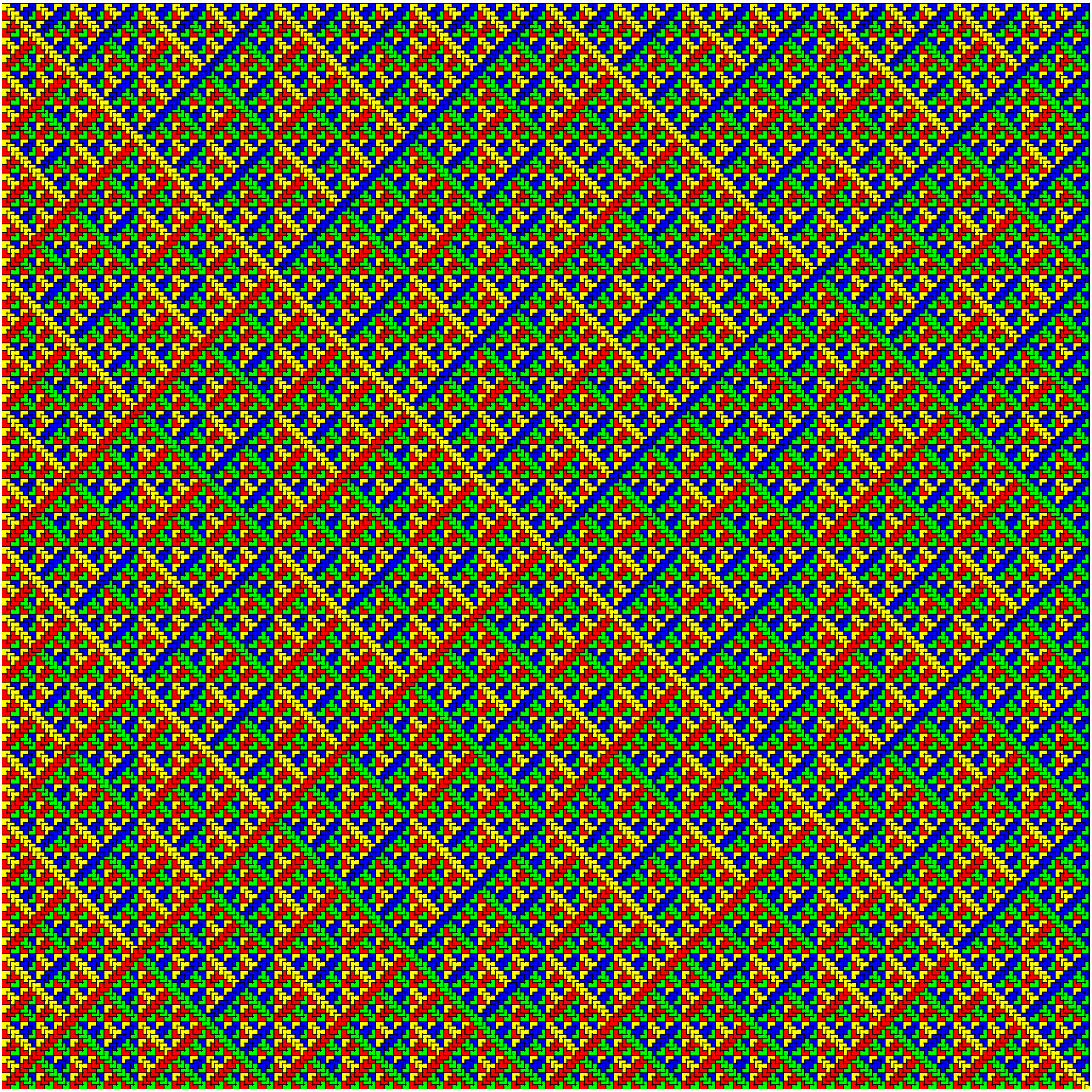

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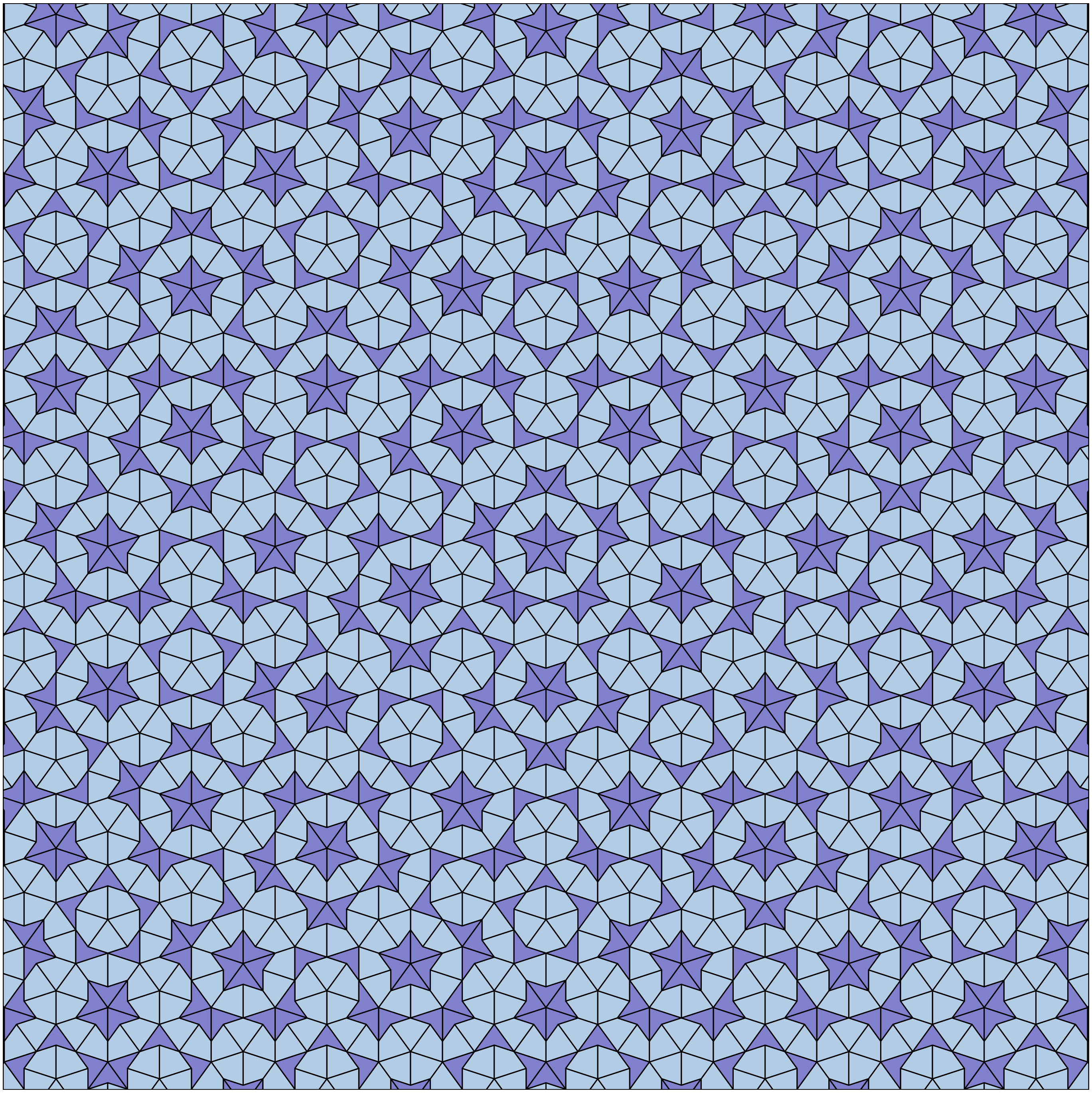
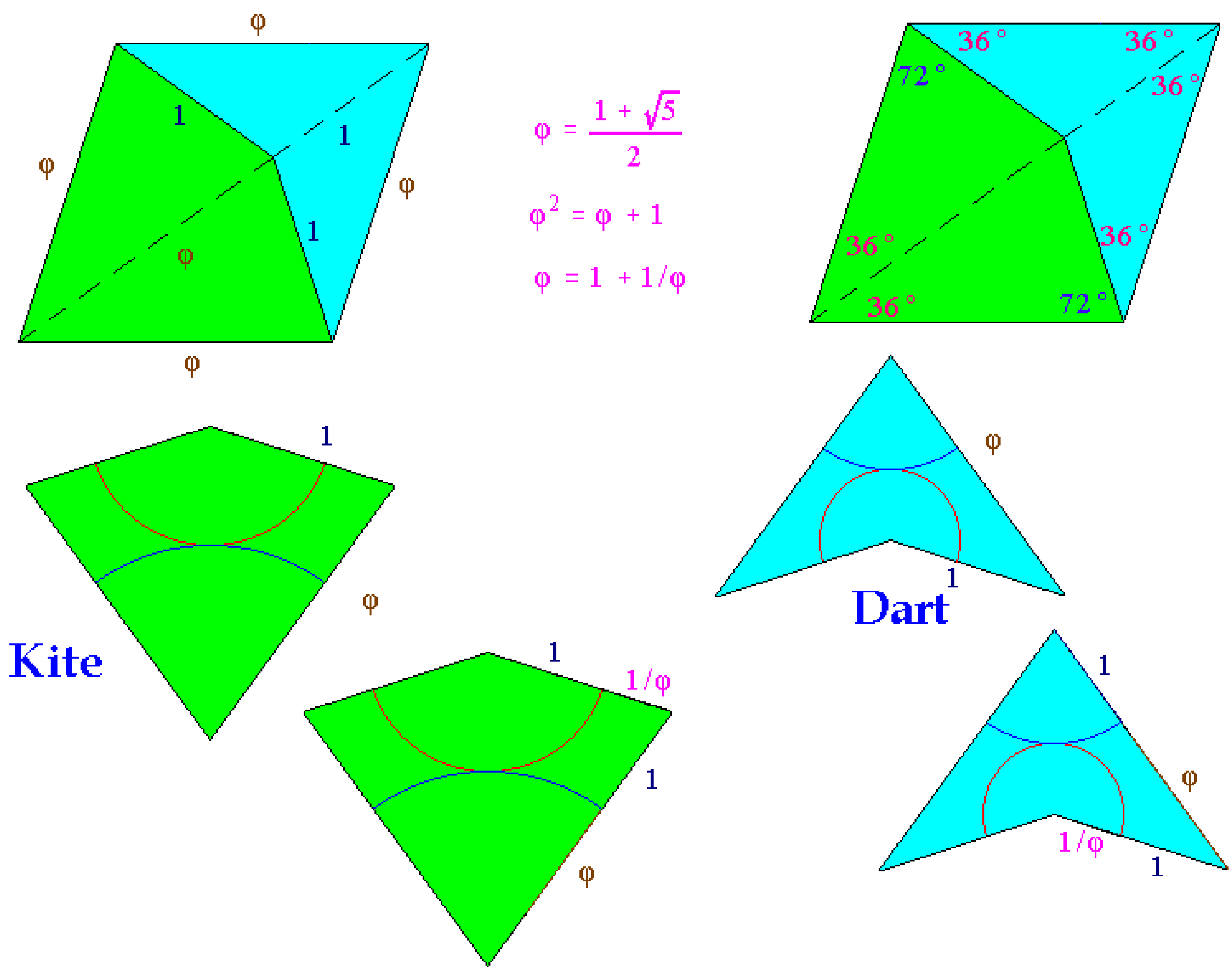
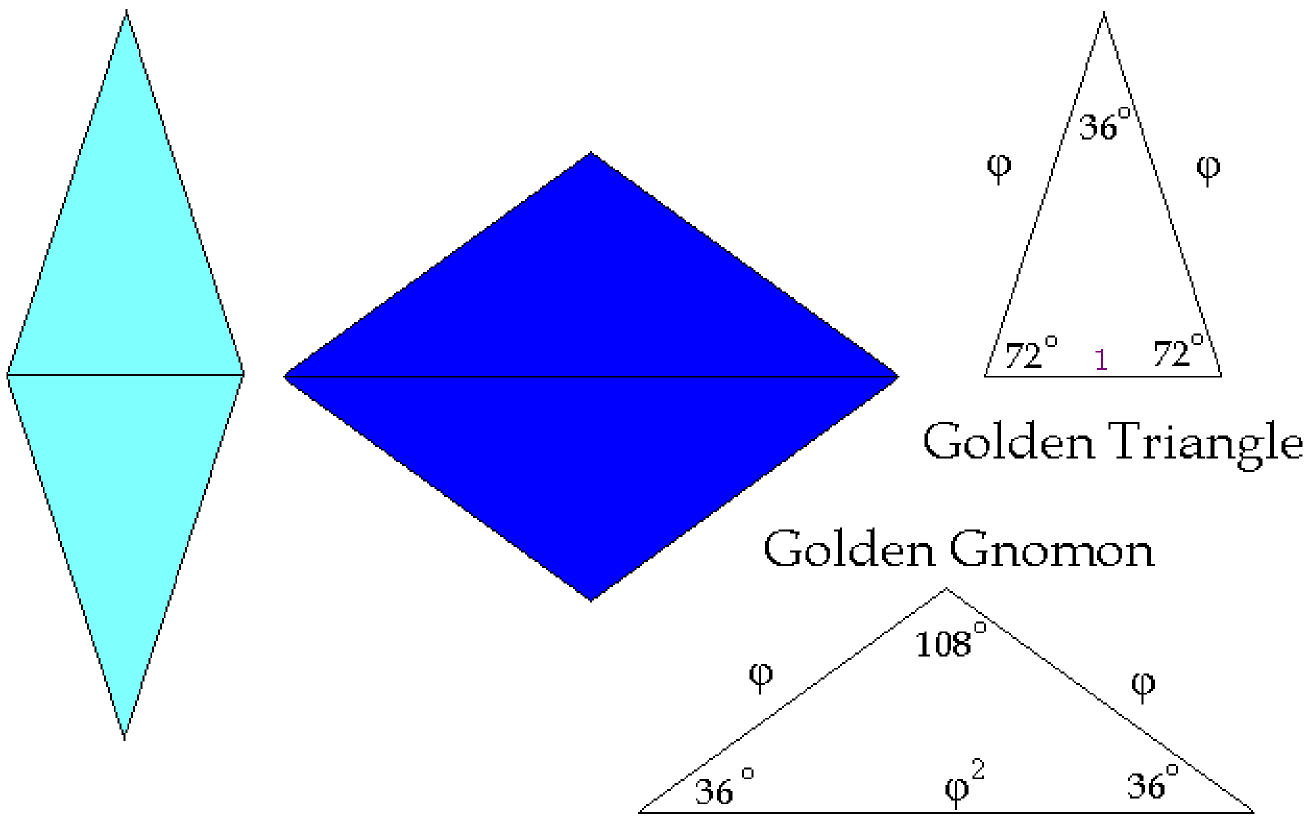


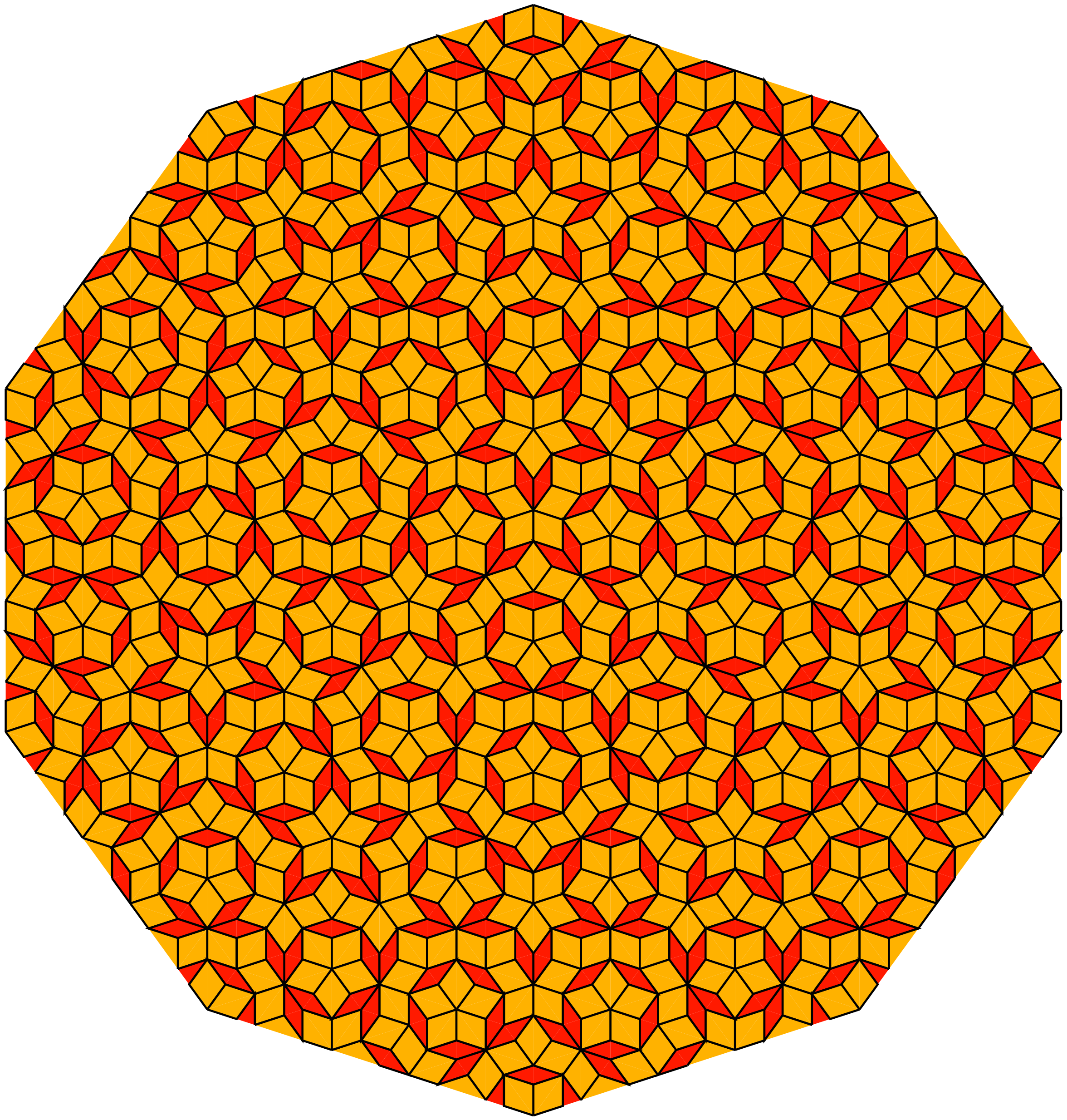
- The Penrose tiling -



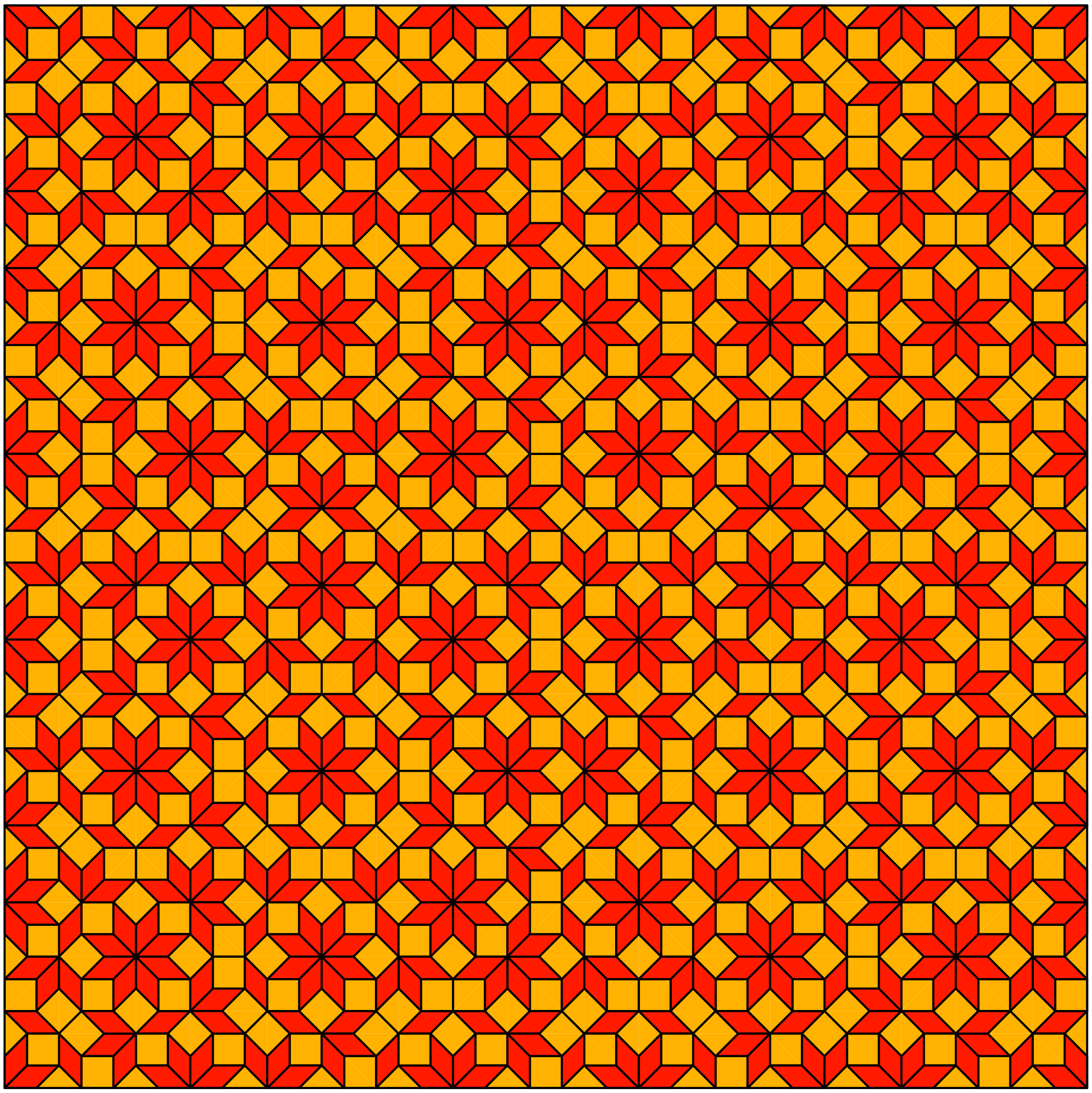
- Kites and Darts -



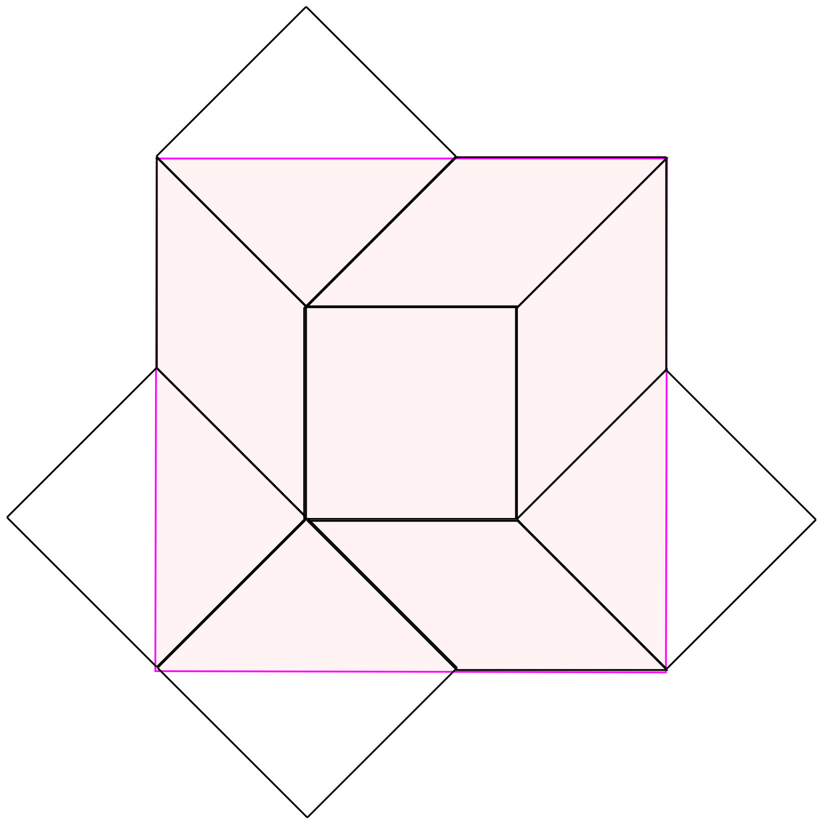
- Rhombi in Penrose's tiling -

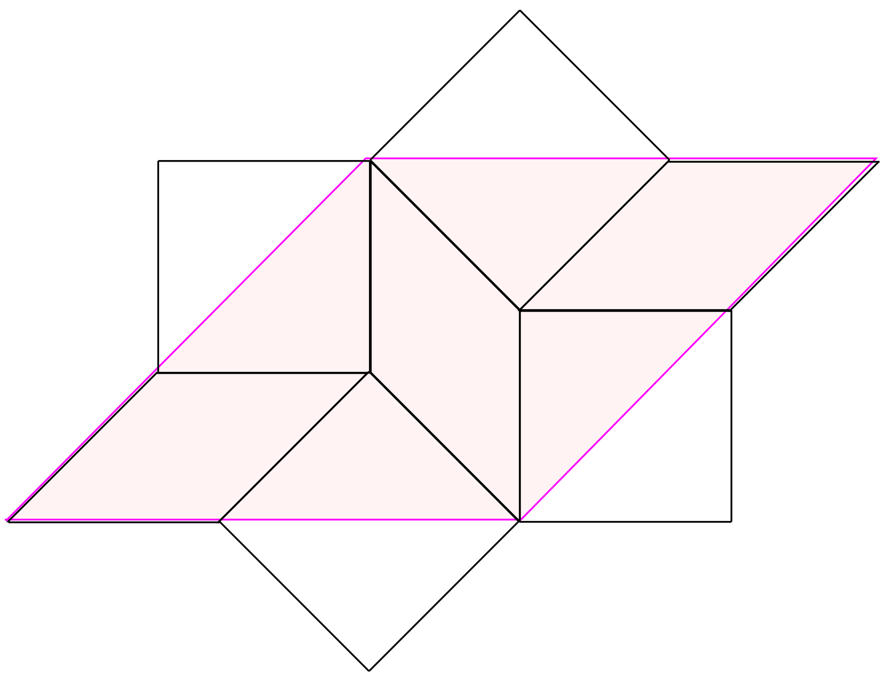


- The Penrose tiling -

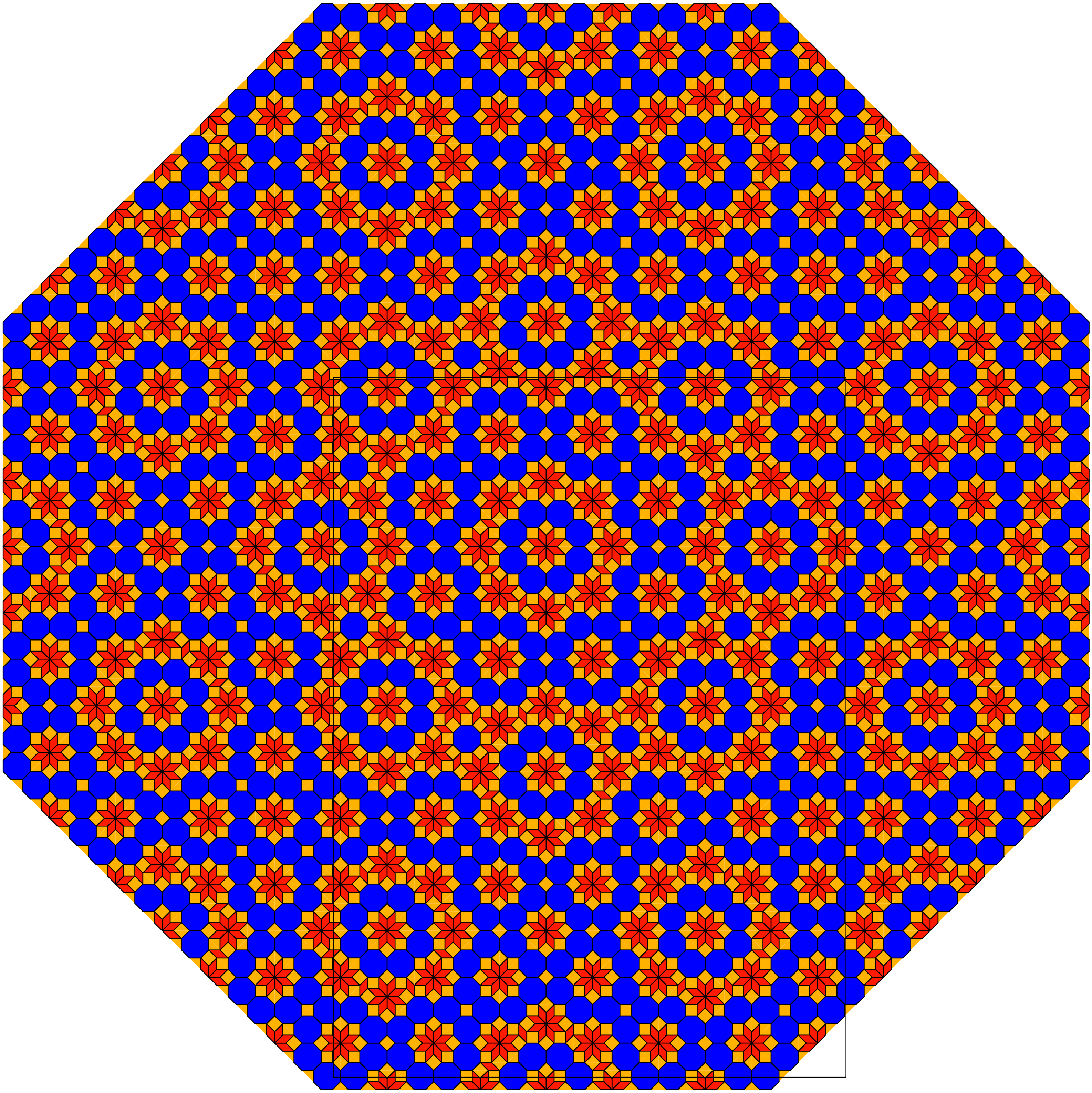


- The octagonal tiling -





- Octagonal tiling: inflation rules -



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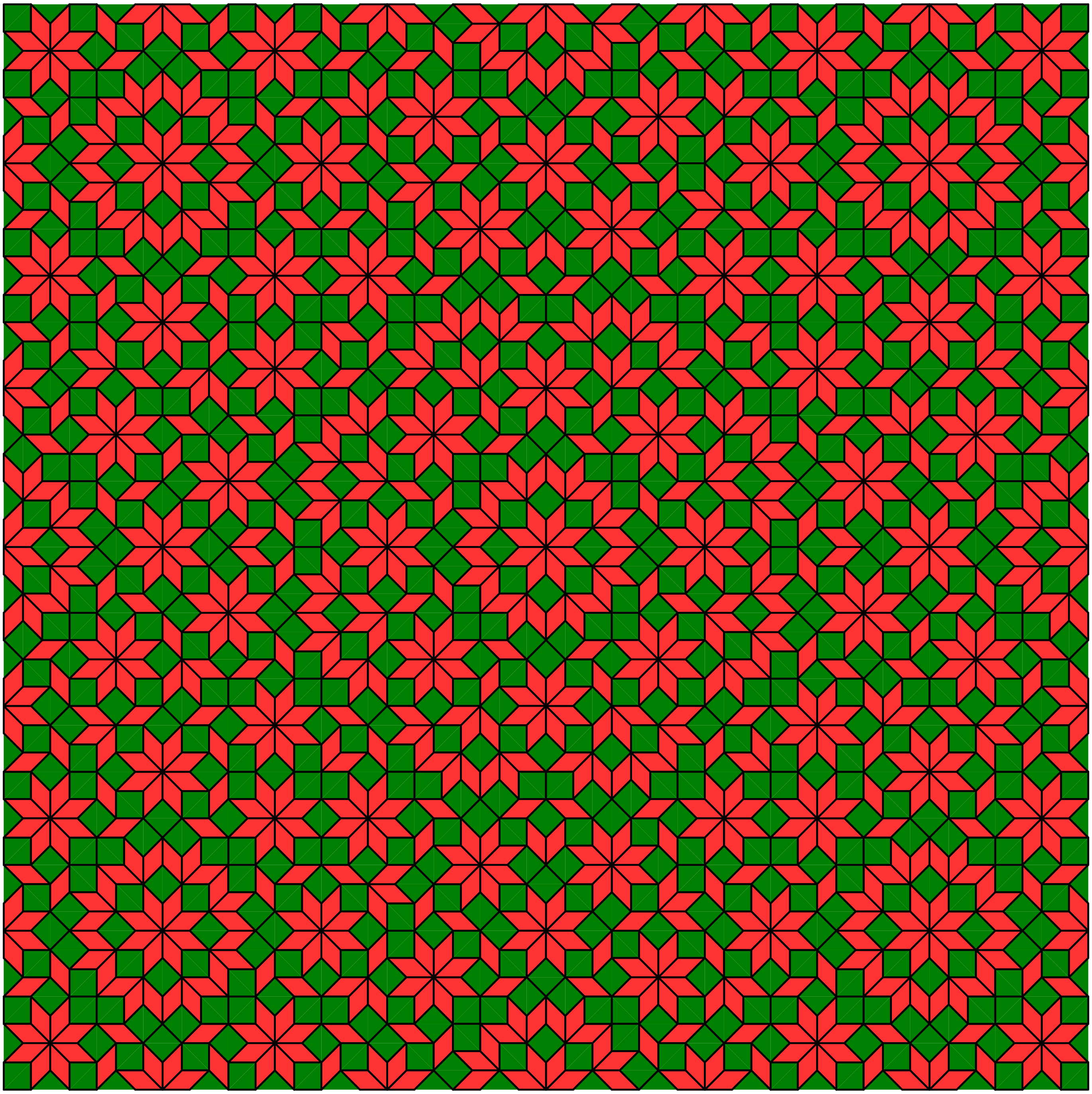

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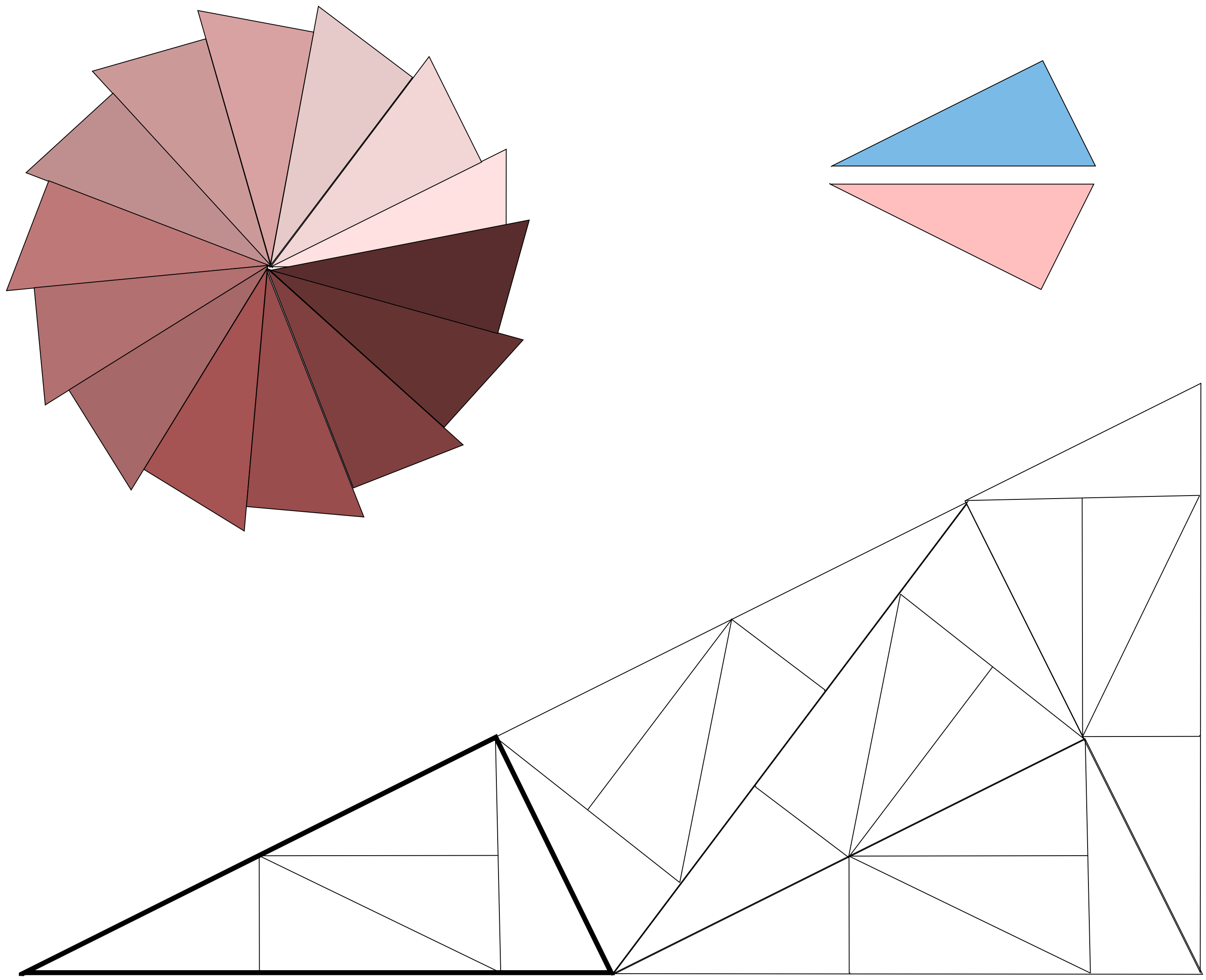
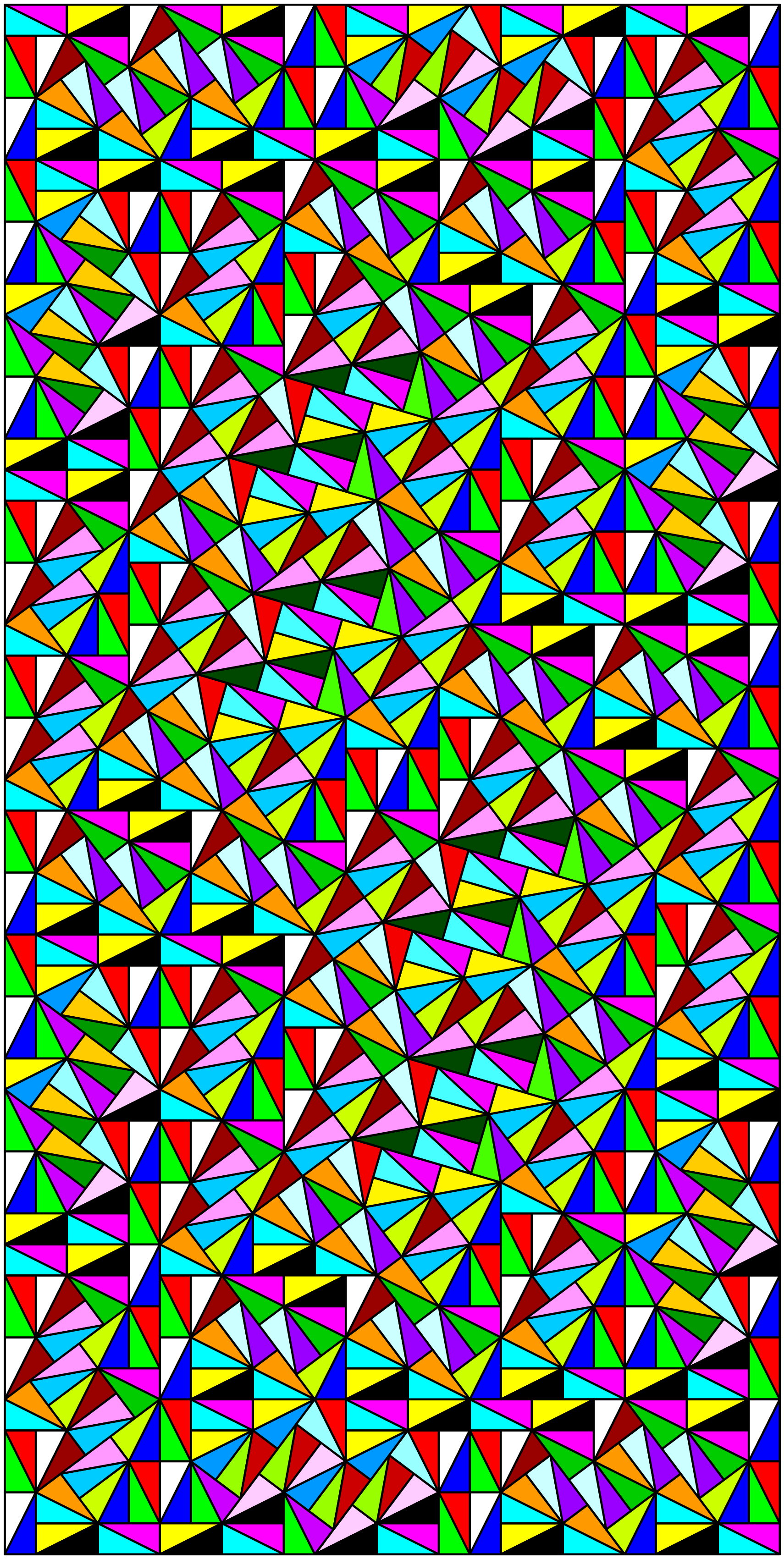
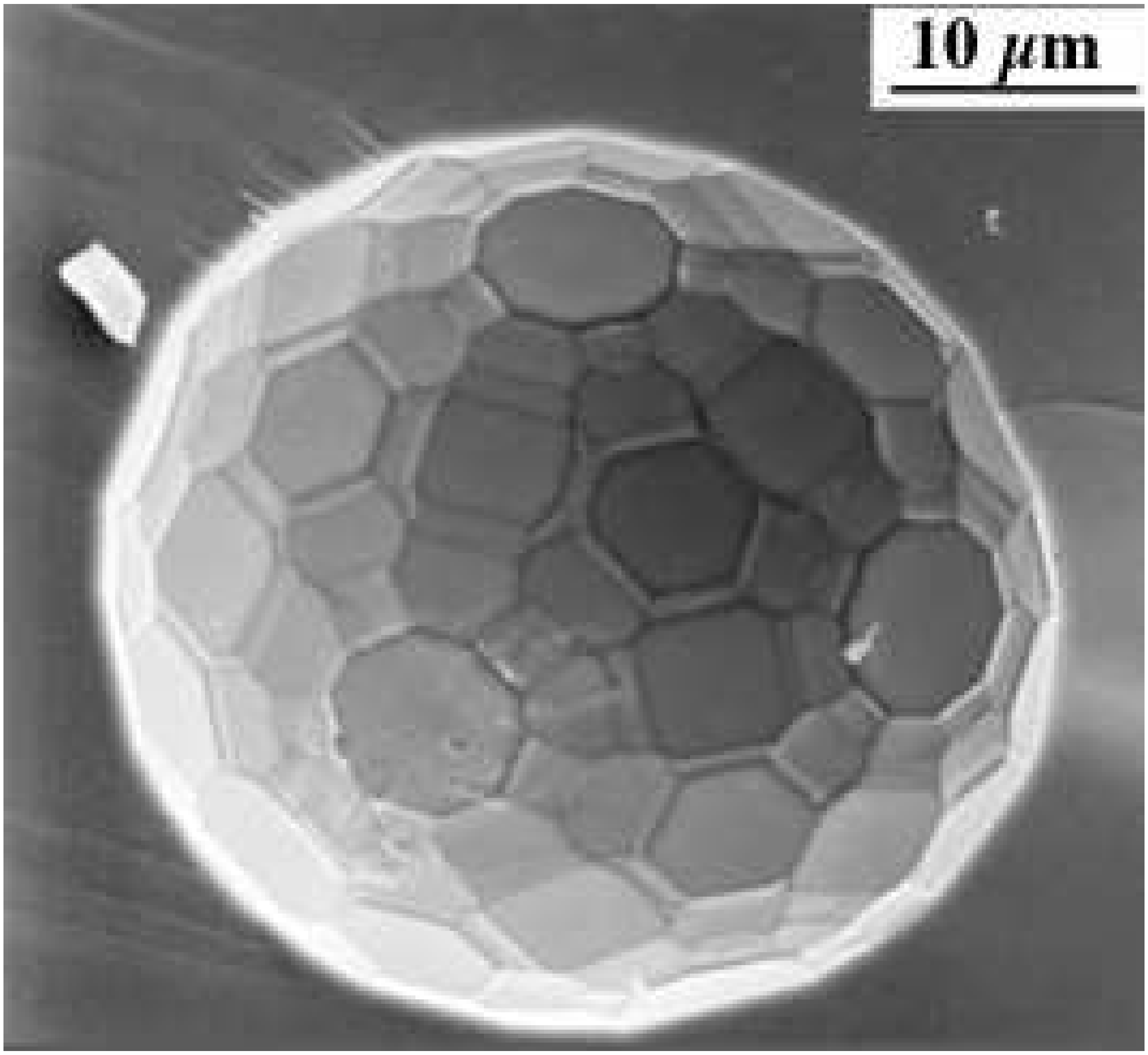


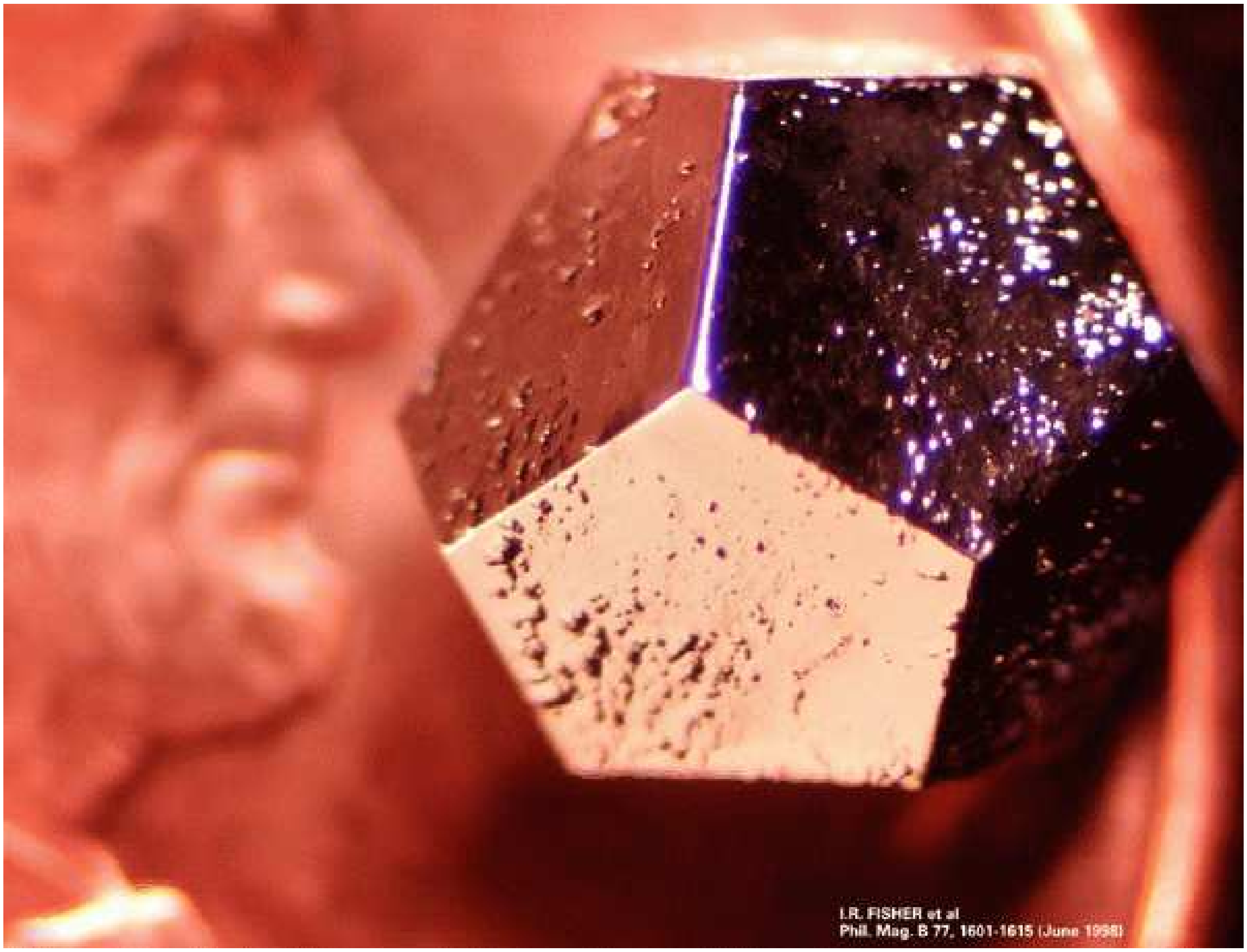
- Building the Pinwheel Tiling -



- The Pinwheel Tiling -



- The icosahedral quasicrystal $AlPdMn$ -



- The icosahedral quasicrystal $HoMgZn$-

II - The Hull as a Dynamical System

Point Sets

A subset $\mathcal{L} \subset \mathbb{R}^d$ may be:

1. *Discrete*.
2. *Uniformly discrete*: $\exists r > 0$ s.t. each ball of radius r contains at most one point of $\mathcal{L}$.
3. *Relatively dense*: $\exists R > 0$ s.t. each ball of radius R contains at least one points of $\mathcal{L}$.
4. A *Delone* set: $\mathcal{L}$ is uniformly discrete and relatively dense.
5. *Finite type Delone* set: $\mathcal{L} - \mathcal{L}$ is discrete.
6. *Meyer* set: $\mathcal{L}$ and $\mathcal{L} - \mathcal{L}$ are Delone.

$\mathfrak{M}(\mathbb{R}^d)$ is the set of Radon measures on $\mathbb{R}^d$ namely the dual space to $\mathcal{C}_c(\mathbb{R}^d)$ (continuous functions with compact support), endowed with the weak* topology.

For $\mathcal{L}$ a *uniformly discrete* point set in $\mathbb{R}^d$:

$$\nu := \nu^{\mathcal{L}} = \sum_{y \in \mathcal{L}} \delta(x - y) \quad \in \mathfrak{M}(\mathbb{R}^d) \ .$$

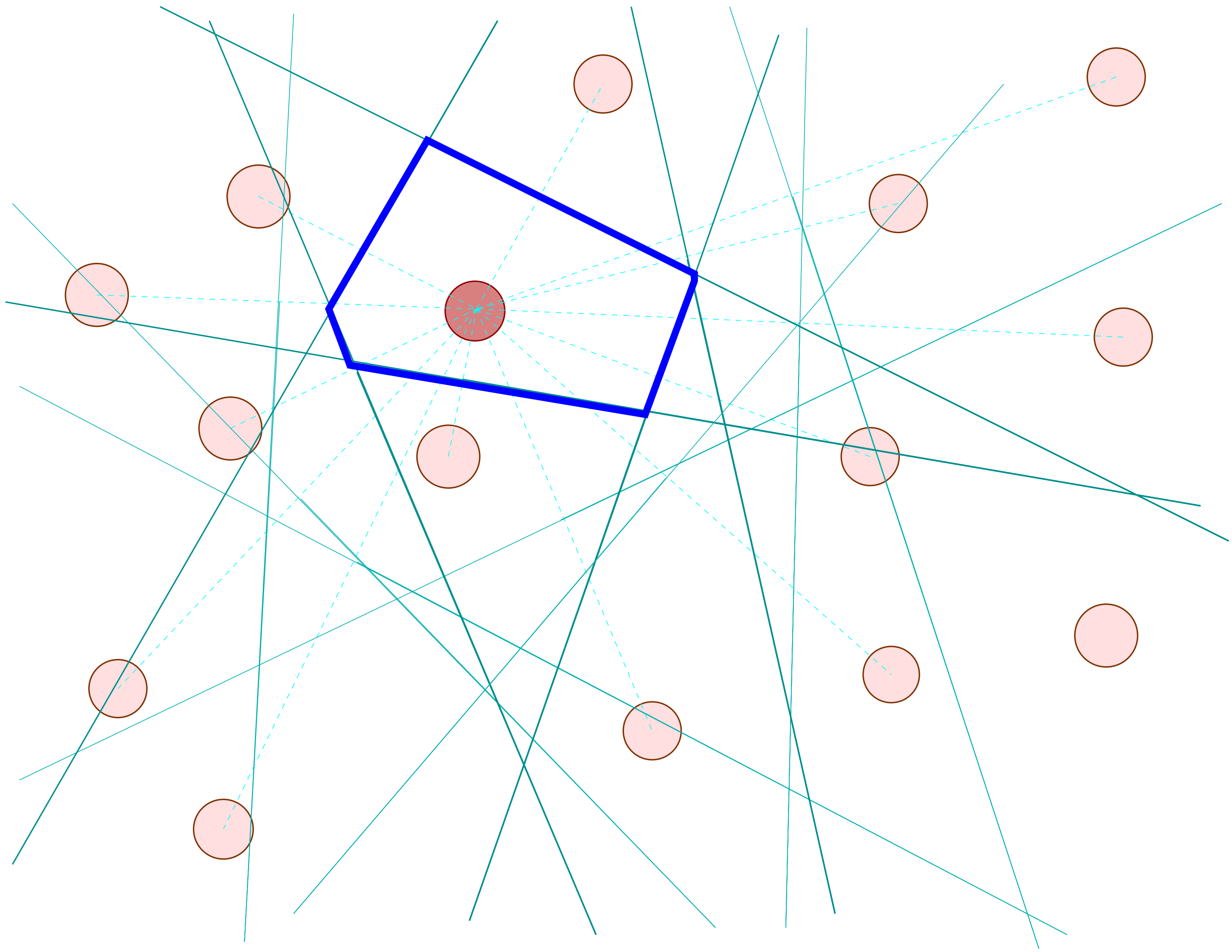
Point Sets and Tilings

Given a tiling with finitely many tiles *(modulo translations)*, a Delone set is obtained by defining a point in the interior of each *(translation equivalence class of)* tile.

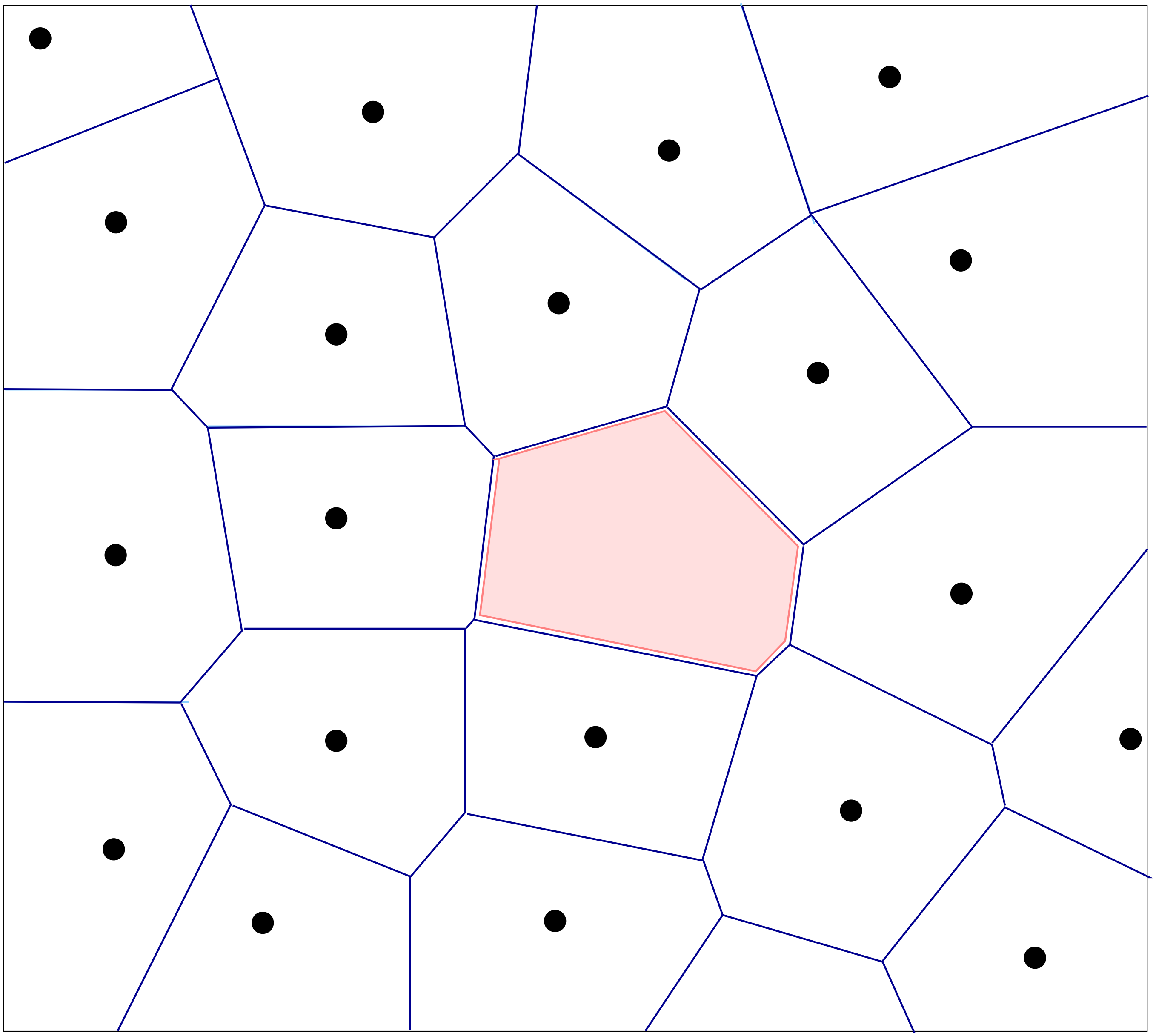
Conversely, given a Delone set, a tiling is built through the *Voronoi cells*

$$V(x) = \{a \in \mathbb{R}^d \,;\, |a-x| < |a-y|\,, \forall y \mathcal{L} \setminus \{x\}\}$$

1. $V(x)$ is an *open convex polyhedron* containing $B(x;r)$ and contained into $\overline{B(x;R)}$.
2. Two Voronoi cells touch face-to-face.
3. If $\mathcal{L}$ has finite type, then the Voronoi tiling has finitely many tiles modulo translations.



- Building a Voronoi cell-



- A Delone set and its Voronoi Tiling-

The Hull

A point measure is $\mu \in \mathfrak{M}(\mathbb{R}^d)$ such that $\mu(B) \in \mathbb{N}$ for any ball $B \subset \mathbb{R}^d$. Its support is

1. *Discrete*.
2. *r-Uniformly discrete*: iff $\forall B$ ball of radius r, $\mu(B) \leq 1$.
3. *R-Relatively dense*: iff for each ball B of radius R, $\mu(B) \geq 1$.

$\mathbb{R}^d$ acts on $\mathfrak{M}(\mathbb{R}^d)$ by translation.

Theorem 1 *The set of r-uniformly discrete point measures is compact and $\mathbb{R}^d$-invariant.*
Its subset of R-relatively dense measures is compact and $\mathbb{R}^d$-invariant.

Definition 1 *Given $\mathcal{L}$ a uniformly discrete subset of $\mathbb{R}^d$, the Hull of $\mathcal{L}$ is the closure in $\mathfrak{M}(\mathbb{R}^d)$ of the $\mathbb{R}^d$-orbit of $\nu^{\mathcal{L}}$.*

Hence the Hull is a *compact metrizable space* on which *$\mathbb{R}^d$ acts by homeomorphisms*.

Properties of the Hull

If $\mathcal{L} \subset \mathbb{R}^d$ is r-uniformly discrete with Hull Ω then using compactness

1. each point $\omega \in \Omega$ *is an r-uniformly discrete* point measure with support $\mathcal{L}_\omega$.
2. if $\mathcal{L}$ is (r, R)-*Delone*, so are all $\mathcal{L}_\omega$'s.
3. if $\mathcal{L}$ has *finite type*, so are all the $\mathcal{L}_\omega$'s.
   Moreover then $\mathcal{L} - \mathcal{L} = \mathcal{L}_\omega - \mathcal{L}_\omega \ \forall \omega \in \Omega$.

Definition 2 *The transversal of the Hull Ω of a uniformly discrete set is the set of $\omega \in \Omega$ such that $0 \in \mathcal{L}_\omega$.*

Theorem 2 *If $\mathcal{L}$ has finite type, then its transversal is completely discontinuous.*

Minimality

A *patch* is a finite subset of $\mathcal{L}$ of the form

$$p = (\mathcal{L} - x) \cap \overline{B(0, r_1)} \qquad x \in \mathcal{L}\,,\ r_1 \geq 0$$

$\mathcal{L}$ is *repetitive* if for any finite patch p there is $R > 0$ such that each ball of radius R contains an ϵ-approximant of a translated of p.

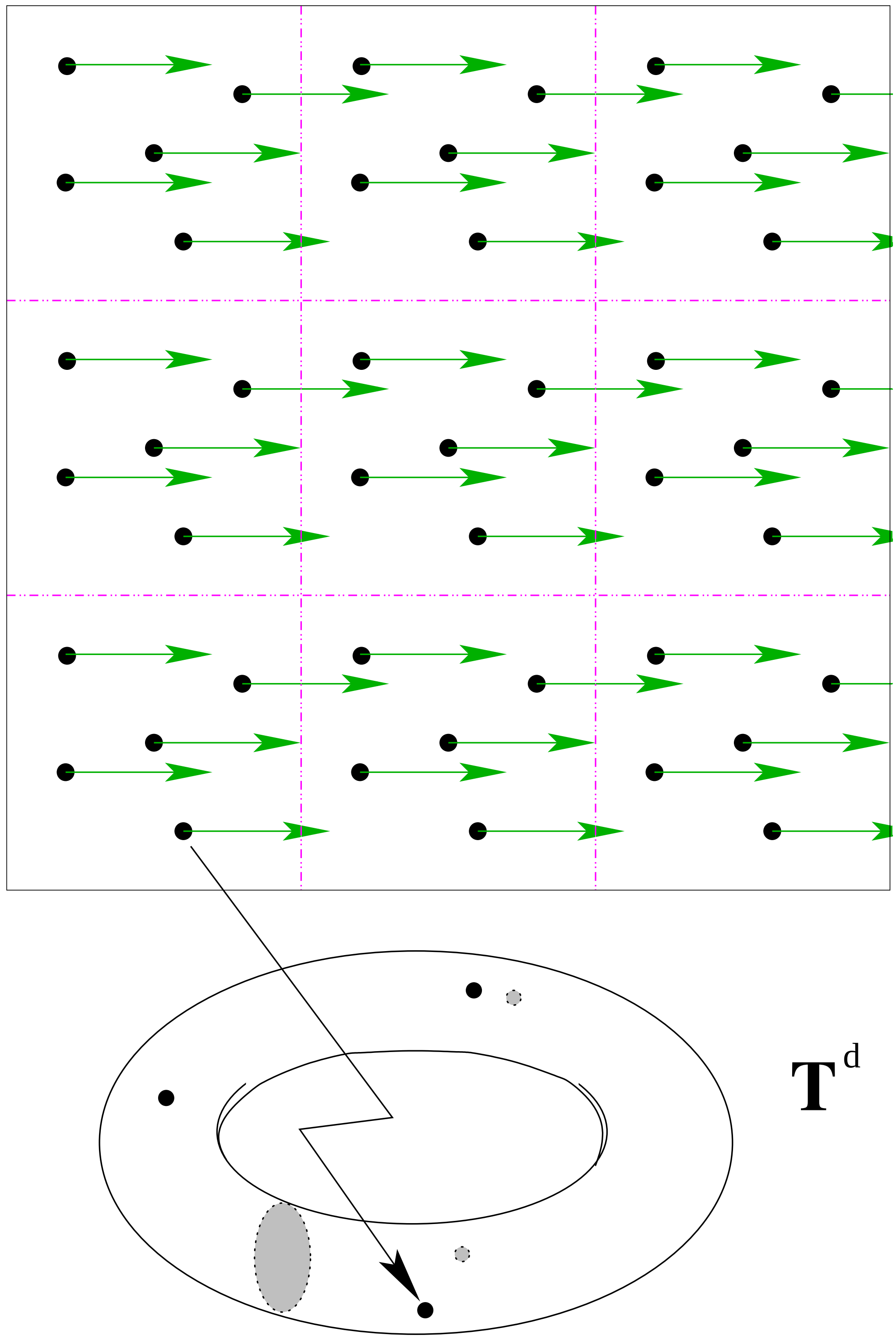
Theorem 3 $\mathbb{R}^d$ *acts minimaly on* Ω *if and only if* $\mathcal{L}$ *is repetitive.*

Examples

1. *Crystals :* $\Omega = \mathbb{R}^d/\mathcal{T} \simeq \mathbb{T}^d$ with the quotient action of $\mathbb{R}^d$ on itself. (Here $\mathcal{T}$ is the translation group leaving the lattice invariant. $\mathcal{T}$ is isomorphic to $\mathbb{Z}^D$.)
   The transversal is a finite set (number of point per unit cell).

2. *Quasicrystals :* $\Omega \simeq \mathbb{T}^n,\ n > d$ with an irrational action of $\mathbb{R}^d$ and a completely discontinuous topology in the transverse direction to the $\mathbb{R}^d$-orbits. The transversal is a Cantor set.

3. *Impurities in* **Si** *:* let $\mathcal{L}$ be the lattices sites for Si atoms (it is a Bravais lattice). Let $\mathfrak{A}$ be a finite set (alphabet) indexing the types of impurities.
   The transversal is $X = \mathfrak{A}^{\mathbb{Z}^d}$ with $\mathbb{Z}^d$-action given by shifts.
   The Hull Ω is the mapping torus of X.



- The Hull of a Periodic Lattice -

Quasicrystals

Use the *cut-and-project* construction:

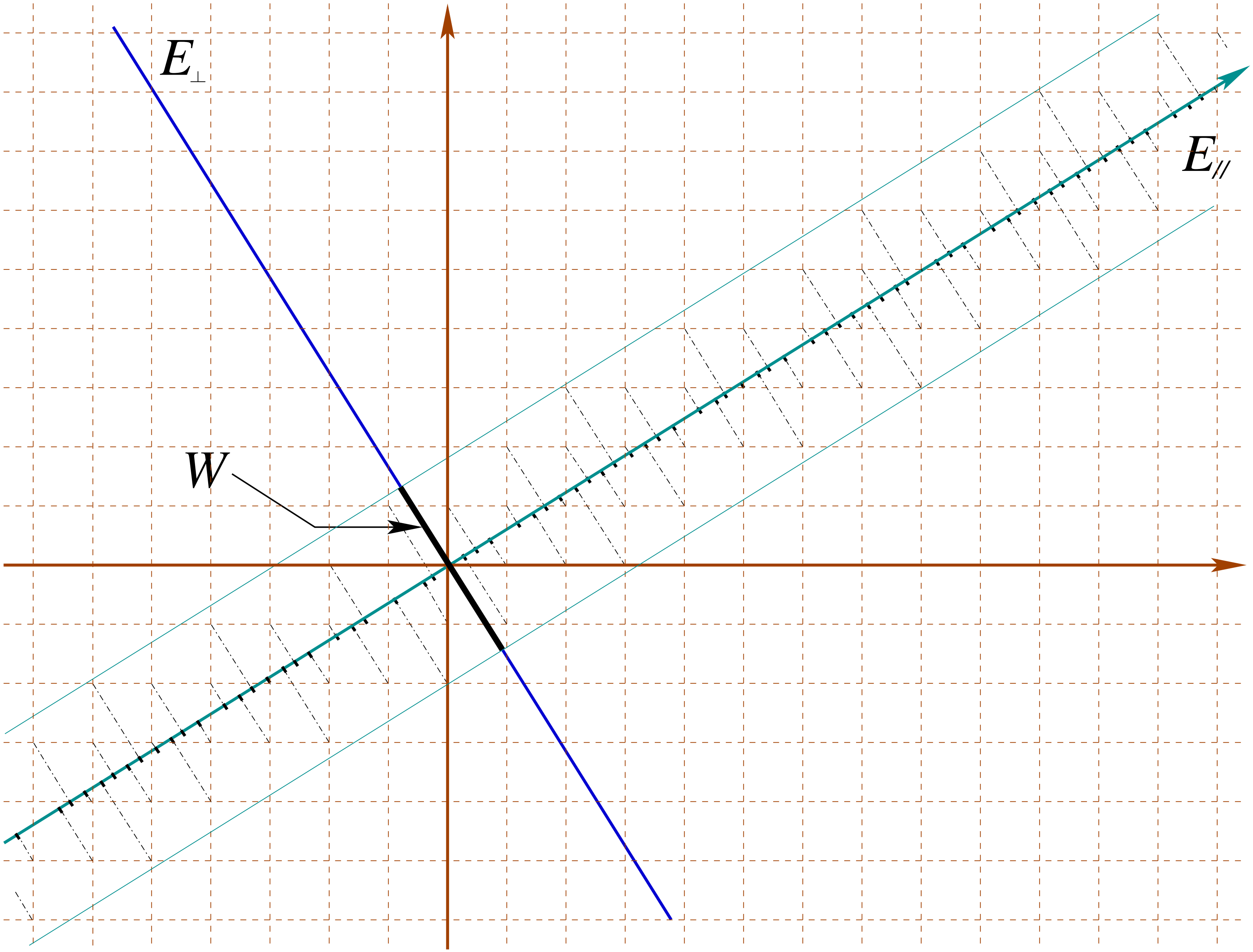
$$\mathbb{R}^d \simeq \mathcal{E}_{\|} \xleftarrow{\pi_{\|}} \mathbb{R}^n \xrightarrow{\pi_{\perp}} \mathcal{E}_{\perp} \simeq \mathbb{R}^{n-d}$$

$$\mathcal{L} \xleftarrow{\pi_{\|}} \tilde{\mathcal{L}} \xrightarrow{\pi_{\perp}} W \quad ,$$

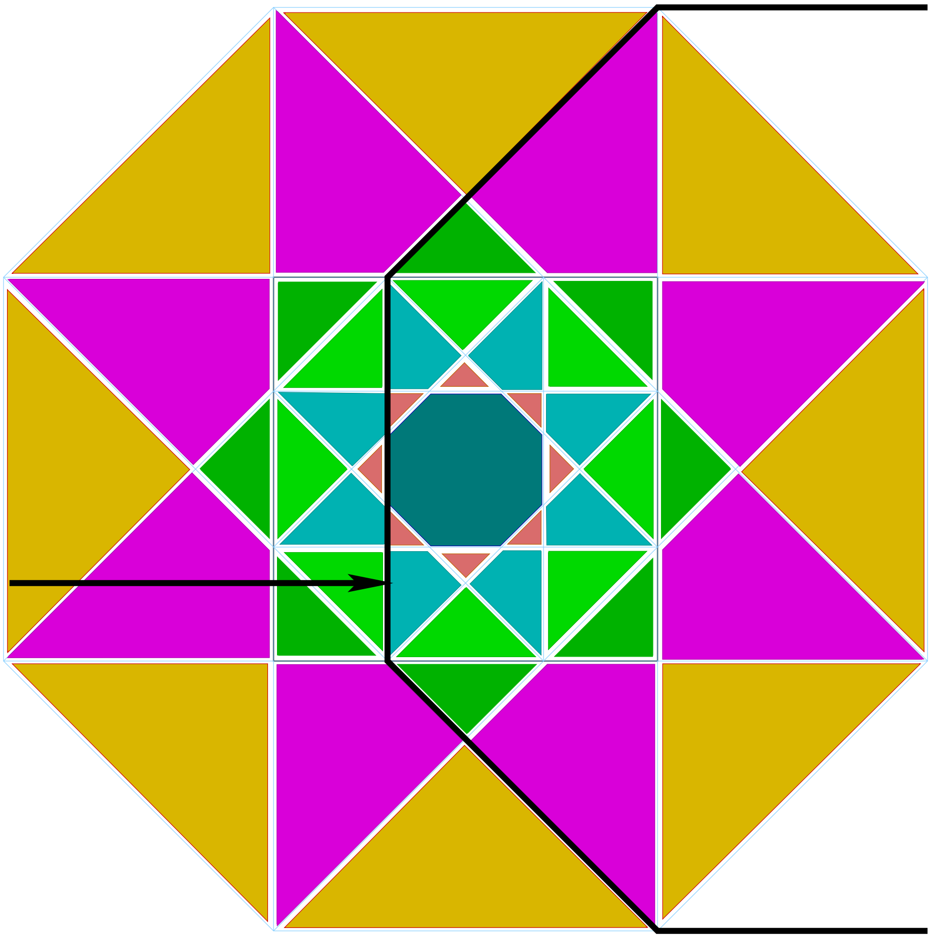
Here

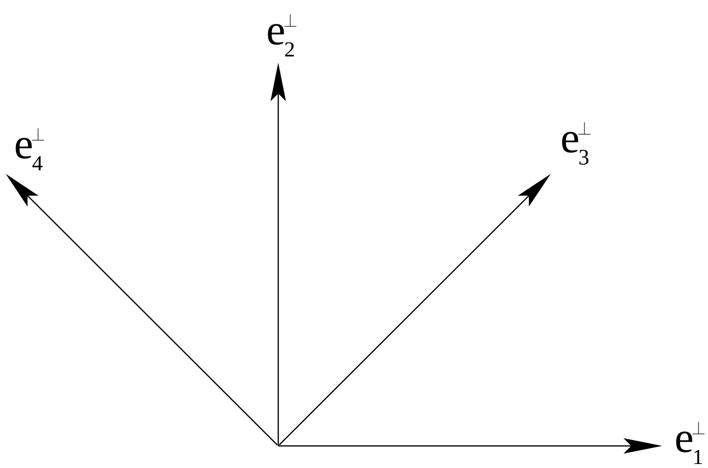
1. $\tilde{\mathcal{L}}$ is a *lattice* in $\mathbb{R}^n$,
2. the *window* W is a compact polytope.
3. $\mathcal{L}$ is the *quasilattice* in $\mathcal{E}_{\|}$ defined as

$$\mathcal{L} \;=\; \{\pi_{\|}(m) \in \mathcal{E}_{\|}\,;\, m \in \tilde{\mathcal{L}}\,,\, \pi_{\perp}(m) \in W\}$$



− The cut−and−project construction −





- The transversal of the Octagonal Tiling -

- is completely disconnected -

III - Branched Oriented Flat Riemannian Manifolds

Laminations and Boxes

A *lamination* is a foliated manifold with $\mathcal{C}^\infty$-structure along the leaves but only finite $\mathcal{C}^0$-structure transversally. The *Hull of a Delone set is a lamination* with $\mathbb{R}^d$-orbits as leaves.

If $\mathcal{L}$ is a *finite type, repetitive, Delone* set, with Hull Ω A *box* is the homeomorphic image of

$$\phi : (\omega, x) \in F \times U \mapsto \mathrm{T}^{-x}\omega \in \Omega$$

if F is a clopen subset of the transversal, $U \subset \mathbb{R}^d$ is open and T denotes the $\mathbb{R}^d$-action on Ω.

A quasi-partition is a family $(B_i)_{i=1}^n$ of boxes such that $\bigcup_i \overline{B_i} = \Omega$ and $B_i \cap B_j = \emptyset$.

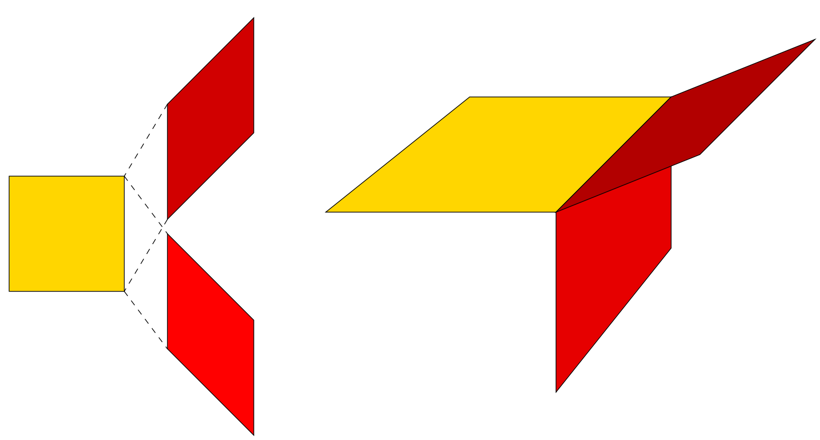
Theorem 4 *The Hull of a finite type, repetitive, Delone set admits a finite quasi-partition. It is always possible to choose these boxes as $\phi(F \times U)$ with U a d-rectangle.*

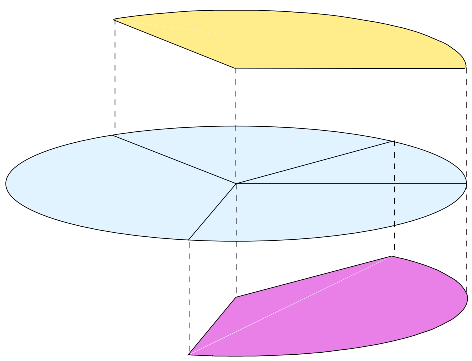
Branched Oriented Flat Manifolds

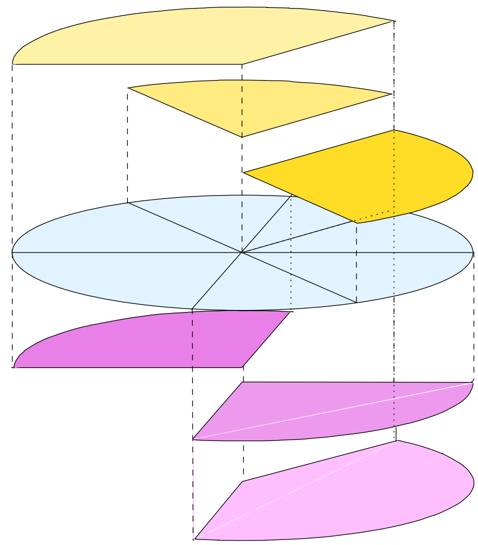
Flattening a box decomposition along the transverse direction leads to a *Branched Oriented Flat manifold*. Such manifolds can be built from the tiling itself as follows

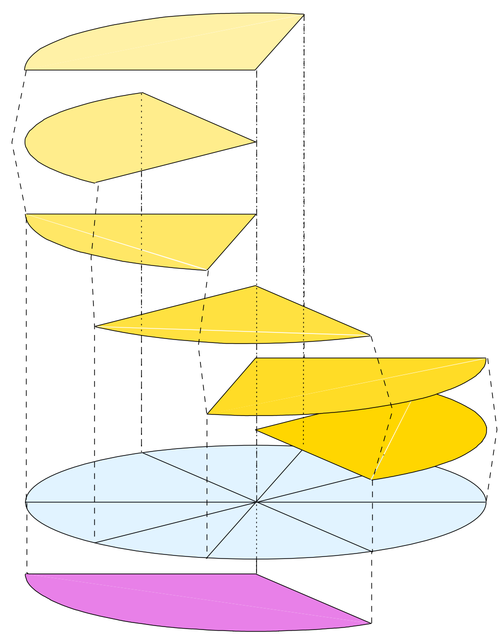
Step 1:

1. X is the disjoint union of all *prototiles*;
2. glue prototiles T_1 and T_2 along a face $F_1 \subset T_1$ and $F_2 \subset T_2$ if F_2 is a translated of F_1 and if there are $x_1, x_2 \in \mathbb{R}^d$ such that $x_i + T_i$ are tiles of $\mathcal{T}$ with $(x_1 + T_1) \cap (x_2 + T_2) = x_1 + F_1 = x_2 + F_2$;
3. after identification of faces, X becomes a *branched oriented flat manifold* (BOF) B_0.









- Vertex branching for the octagonal tiling -

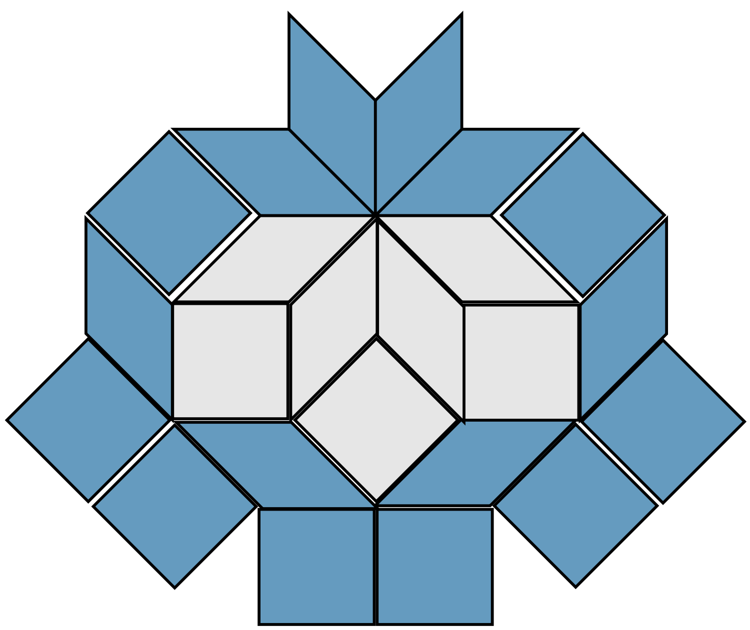
Step 2:

1. Having defined the patch p_n for $n \geq 0$, let $\mathcal{L}_n$ be the subset of $\mathcal{L}$ of points centered at a translated of p_n. By repetitivity this is a finite type repetitive Delone set too. Its prototiles are tiled by tiles of $\mathcal{L}$ and define a finite family $\mathfrak{P}_n$ of patches.

2. Color each patch in $\mathcal{T} \in \mathfrak{P}_n$ by the tiles touching it from outside along its frontier. Call such a patch *modulo translation* a *a colored patch* and $\mathfrak{P}_n^c$ their set.

3. Proceed then as in Step 1 by replacing prototiles by colored patches to get the BOF-manifold B_n.

4. Then choose for p_{n+1} as the colored patch in $\mathfrak{P}_n^c$ containing p_n.



Step 3:

1. Define a *BOF-submersion* $f_n : B_{n+1} \mapsto B_n$ by identifying patches of order n in B_{n+1} with the prototiles of B_n. Note that $Df_n = \mathbf{1}$.

2. Call Ω the *projective limit* of the sequence

$$\cdots \xrightarrow{f_{n+1}} B_{n+1} \xrightarrow{f_n} B_n \xrightarrow{f_{n-1}} \cdots$$

3. $X_1, \cdots X_d$ are the commuting constant vector fields on B_n generating local translations and giving rise to a $\mathbb{R}^d$ action T on Ω.

Theorem 5 *The dynamical system*

$$(\Omega, \mathbb{R}^d, \mathrm{T}) = \varprojlim (B_n, f_n)$$

obtained as inverse limit of branched oriented flat manifolds, is conjugate to the Hull of the Delone set of the tiling $\mathcal{T}$ by an homemorphism.

Longitudinal (co)-Homology

The Homology groups are defined by the inverse limit

$$H_*(\Omega, \mathbb{R}^d) \;=\; \lim_{\leftarrow} \left(H_*(B_n, \mathbb{R}), f_n^*\right)$$

Theorem 6 *The homology group $H_d(\Omega, \mathbb{R}^d)$ admits a canonical positive cone induced by the orientation of $\mathbb{R}^d$, isomorphic to the affine set of positive $\mathbb{R}^d$-invariant measures on Ω.*

The cohomology groups are defined by the direct limit

$$H^*(\Omega, \mathbb{R}^d) \;=\; \lim_{\rightarrow} \left(H^*(B_n, \mathbb{R}), f_n^*\right)$$

Theorem 7 *If $\mathbb{P}$ is an $\mathbb{R}^d$-invariant probability on Ω, then the pairing with $H^d(\Omega, \mathbb{R}^d)$ satisfies*

$$\langle \mathbb{P} | H^d(\Omega, \mathbb{R}^d) \rangle \;=\; \int_\Xi d\mathbb{P}_{\mathrm{tr}} \; \mathcal{C}(\Xi, \mathbb{Z})$$

where Ξ is the transversal, $\mathbb{P}_{\mathrm{tr}}$ is the probability on Ξ induced by $\mathbb{P}$ and $\mathcal{C}(\Xi, \mathbb{Z})$ is the space of integer valued continuous functions on Ξ.

Conclusion

1. *Tilings* can be equivalently be represented by *Delone sets* or *point measures*.
2. The *Hull* allows to give tilings the structure of a *dynamical system* with a transversal.
3. This dynamical system can be seen as a *lamination* and admits *finite box decompositions*.
4. Such box decompositions gives rise to *inverse limits* of *Branched Oriented Flat Riemannian Manifolds*, an equivalent description.
5. Such inverse limit gives the *longitudinal Homology* and *Cohomology* groups of this tiling space. In maximum degree, they give the family of *invariant measures* and the *Gap Labelling Theorem*.