ATOMIC MOTION

and

TRANSVERSE GEOMETRY

of

TILING SPACES

Sponsoring

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Main References

A. Beurling, J. Deny, *Dirichlet spaces*, Proc. Nat. Acad. Sci. U.S.A., **45**, (1959), 208-215.

M. Fukushima, *Dirichlet forms and Markov processes*,
North-Holland Math. Lib., 23., Amsterdam-New York; Kodansha, Ltd., Tokyo, 1980.

A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

G. Michon, *Les Cantors réguliers*, C. R. Acad. Sci. Paris Sér. I Math., (19), **300**, (1985) 673-675.

J. Pearson, J. Bellissard, *Noncommutative Riemannian Geometry and Diffusion on Ultrametric Cantor Sets*,
J. Noncommutative Geometry, **3**, (2009), 447-480.

A. Julien, J. Savinien, *Transverse Laplacians for substitution tilings*,
Comm. Math. Phys., **301**, (2011), 285-318.

I. Palmer, *Noncommutative Geometry and Compact Metric Spaces*,
PhD Thesis, Georgia Institute of Technology, May 2010

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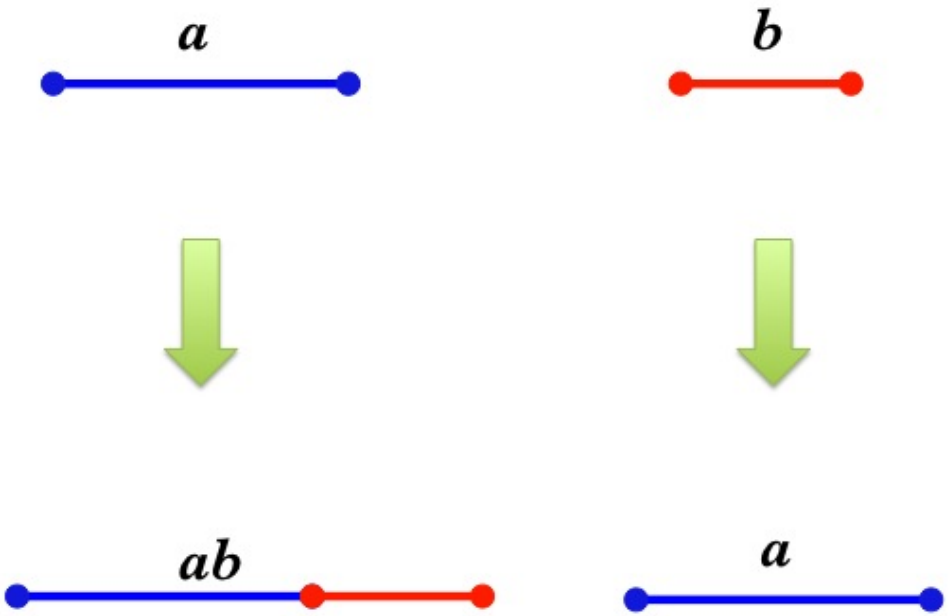
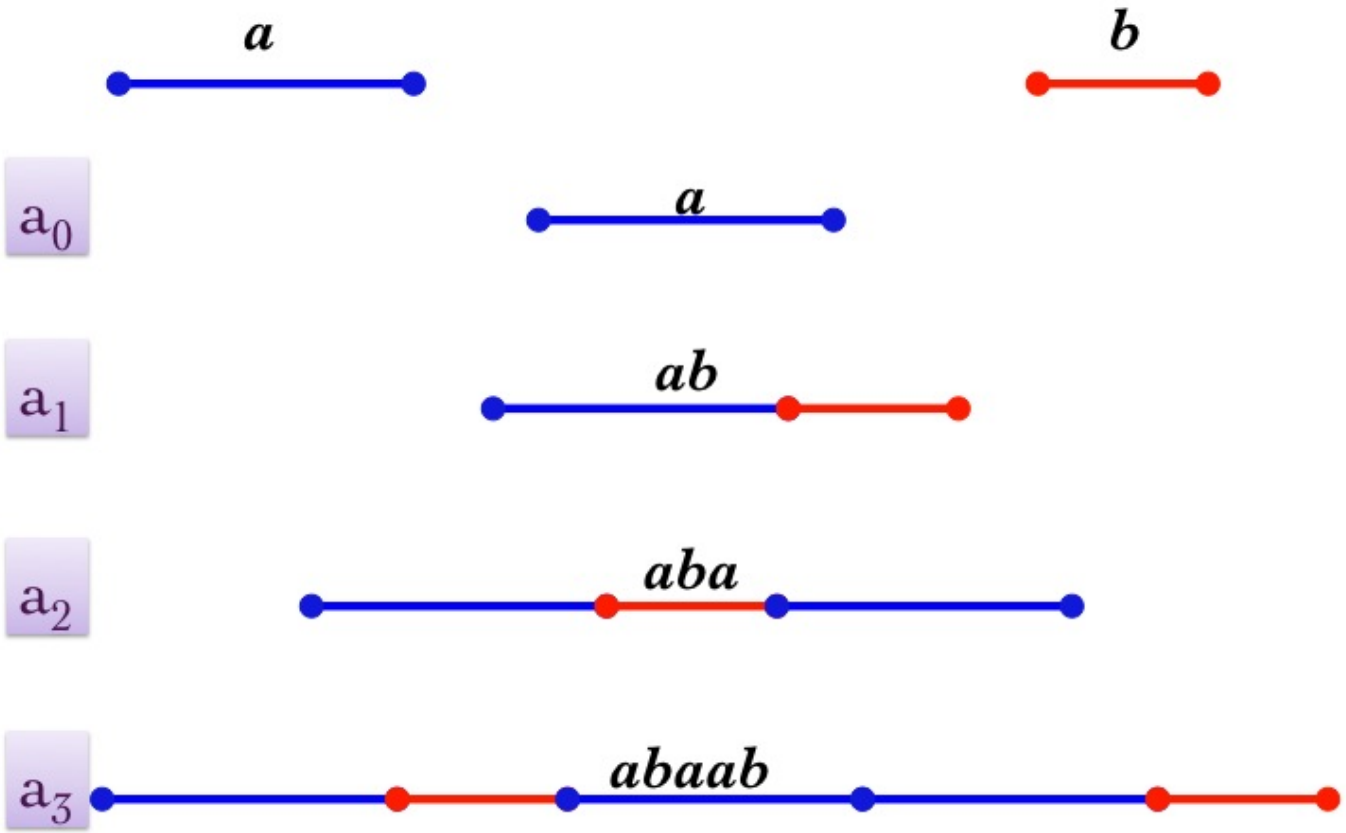
I - Tilings and Flip-Flops

The Fibonacci Tiling



The Fibonacci Substitution

The Fibonacci Tiling



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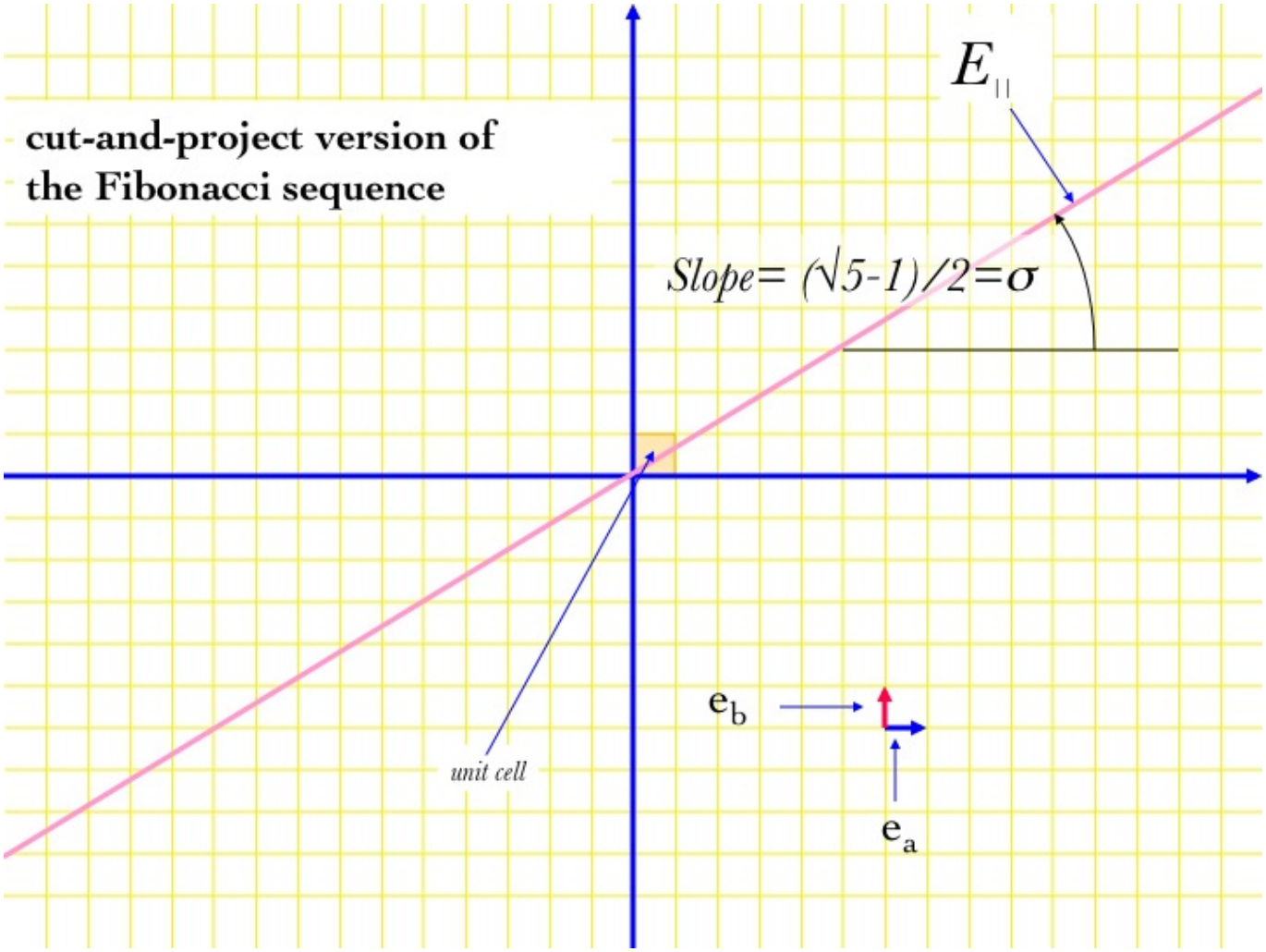

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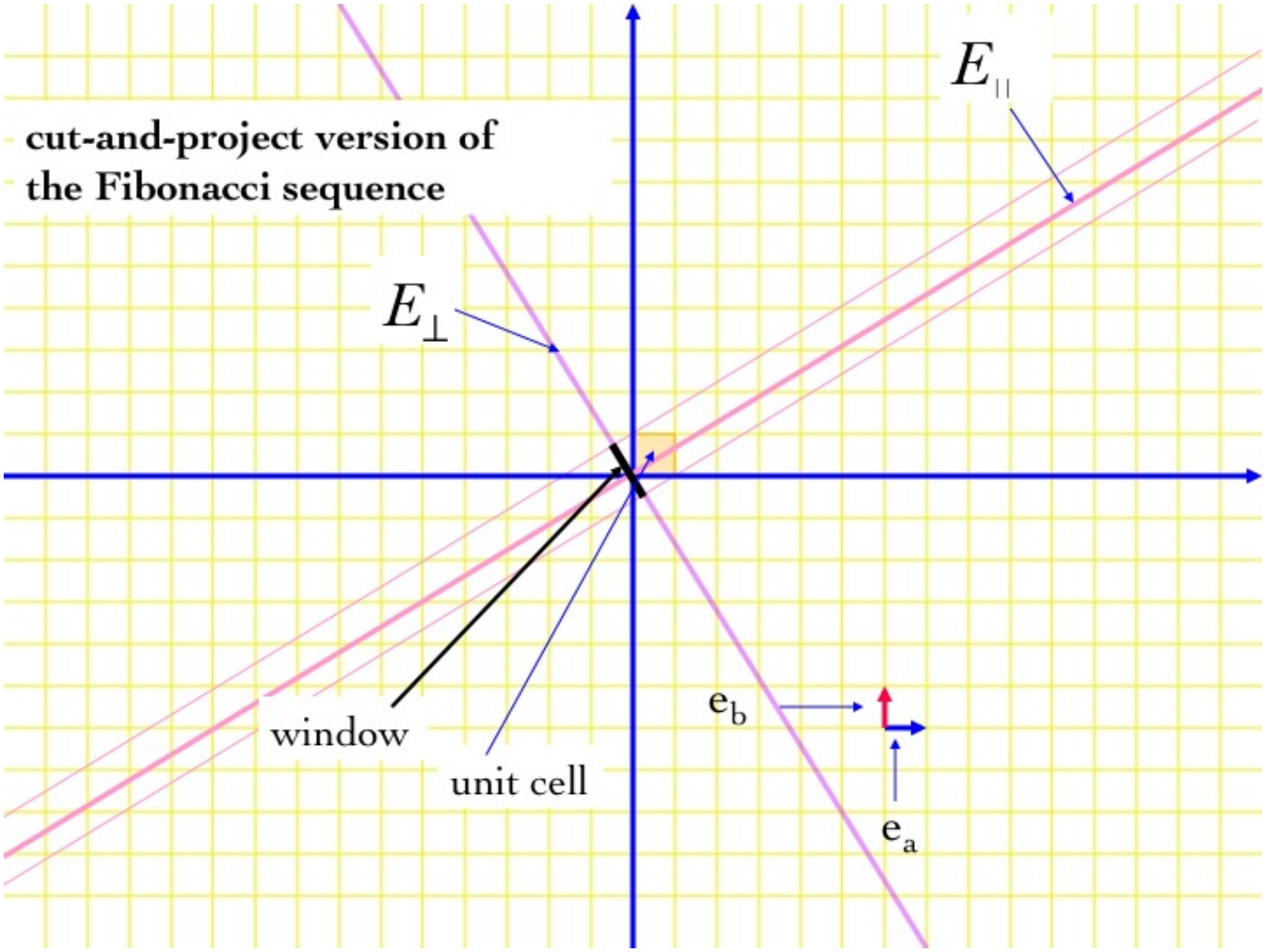

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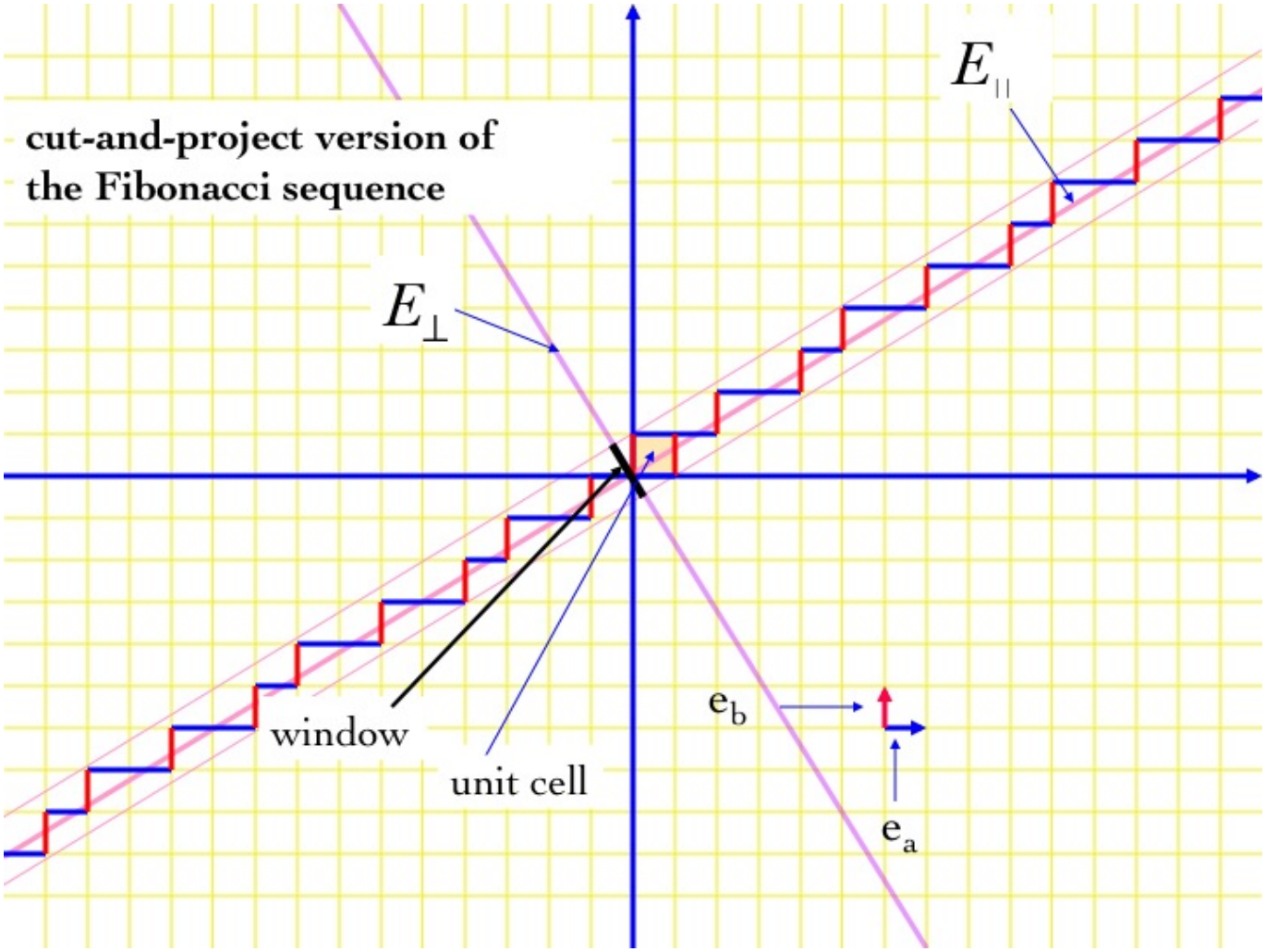

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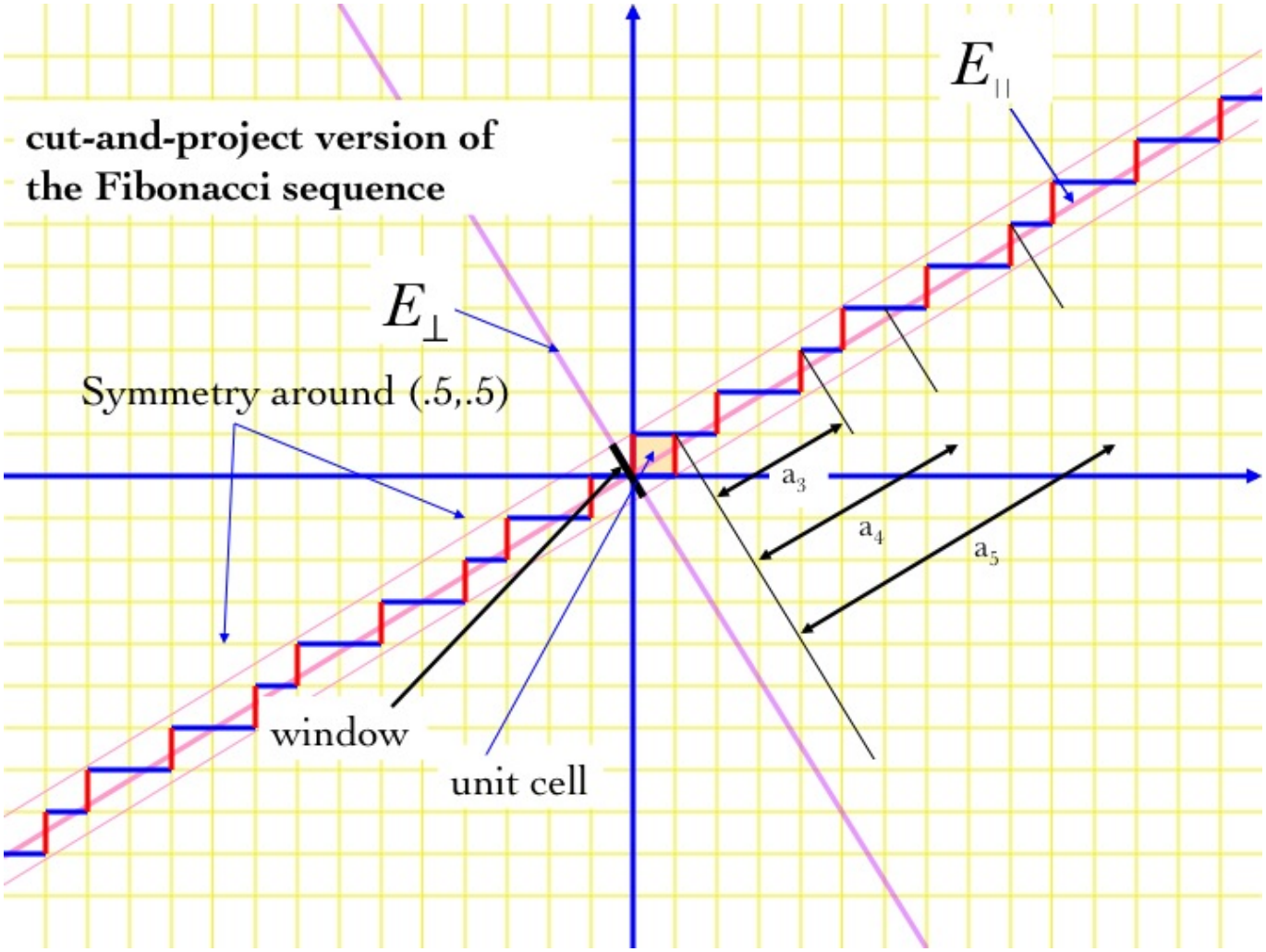

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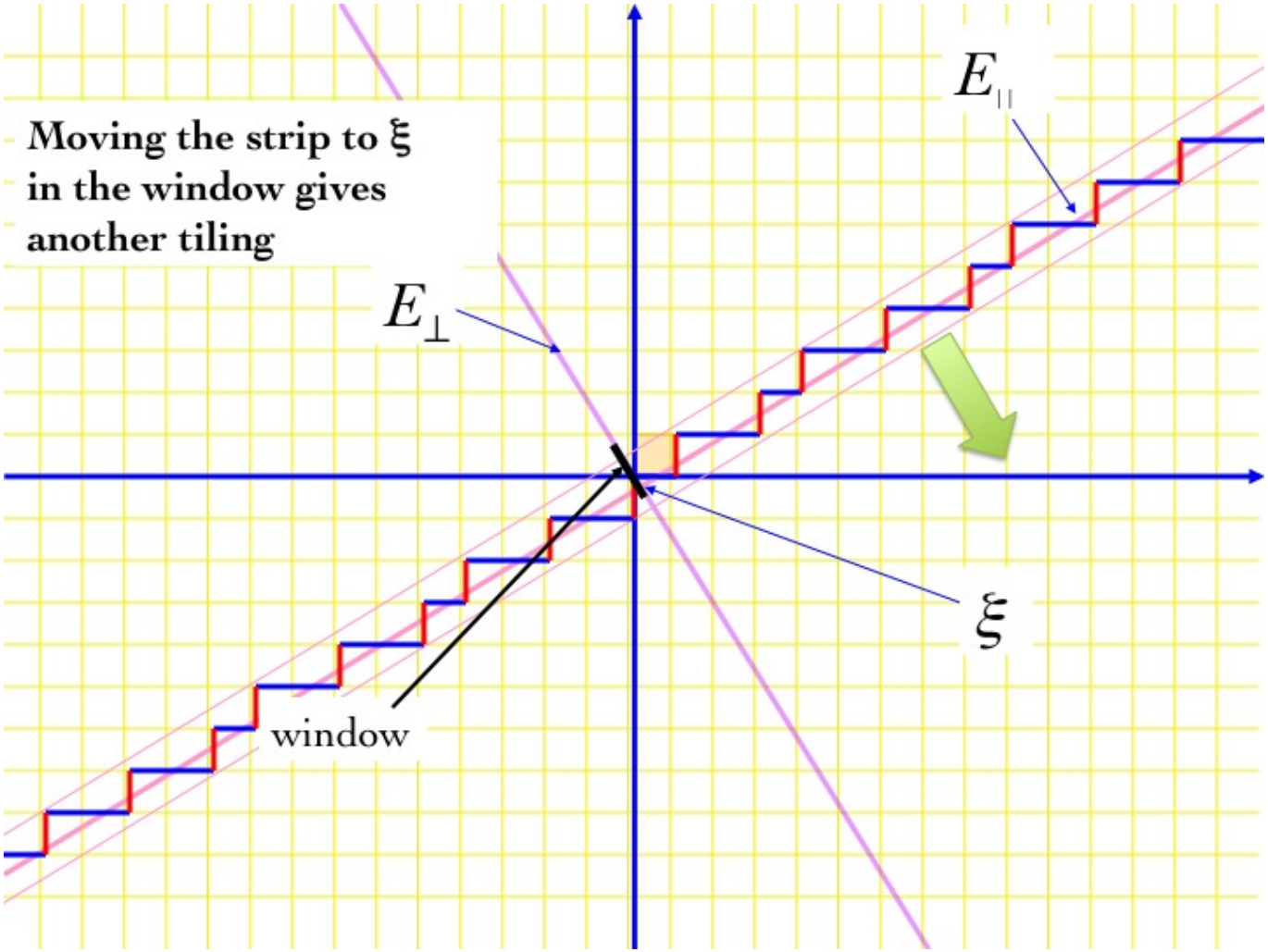

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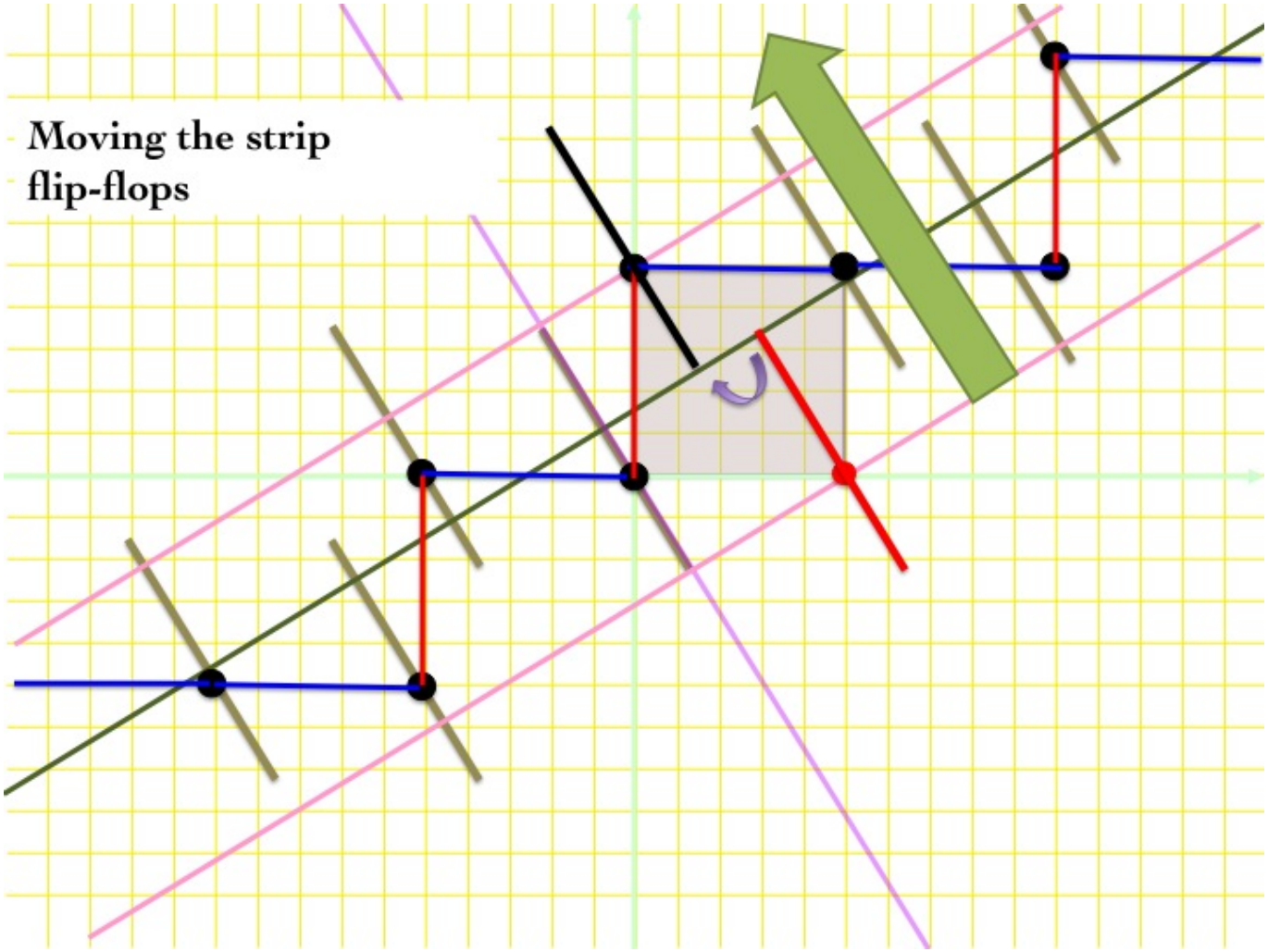

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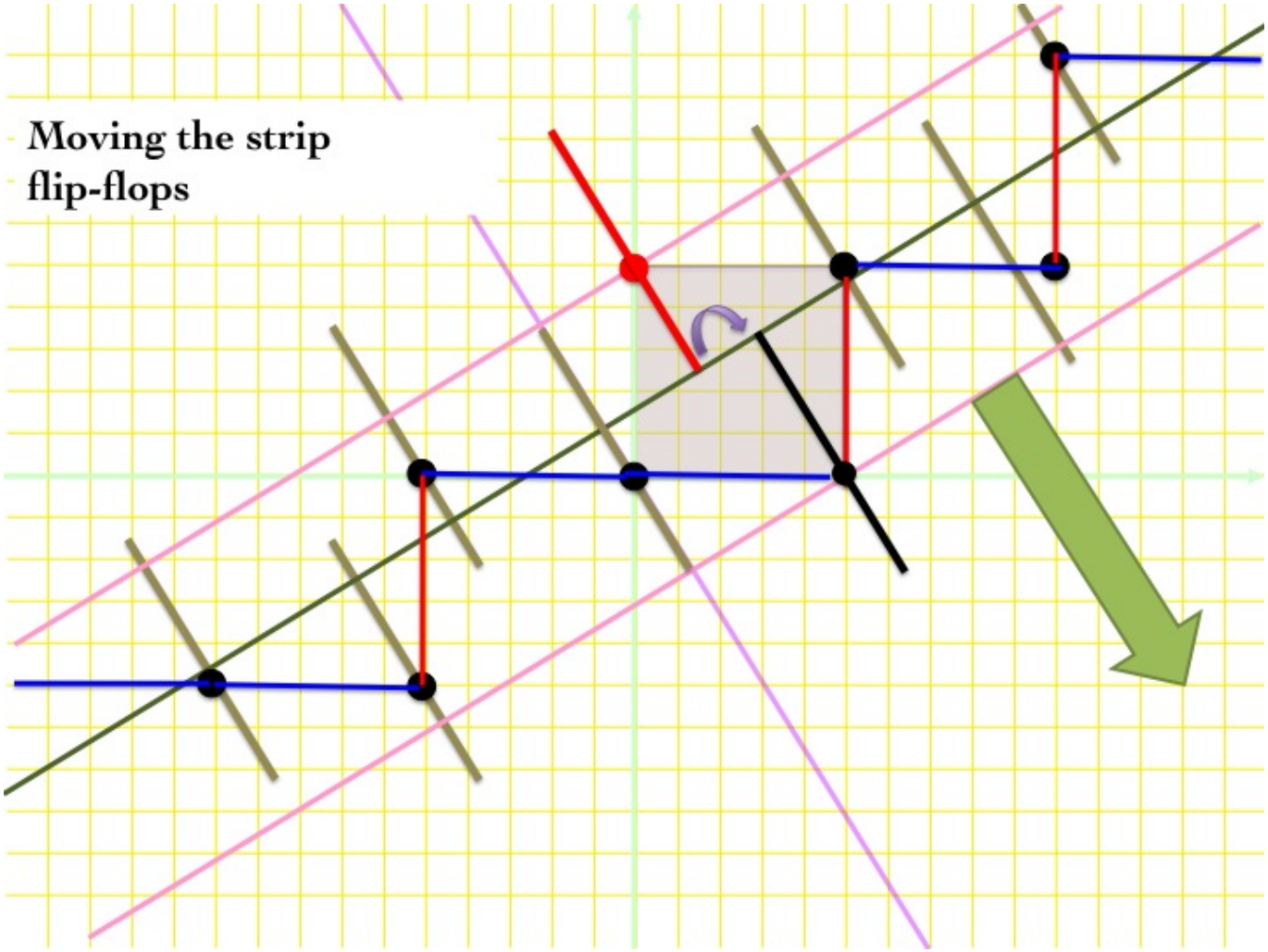

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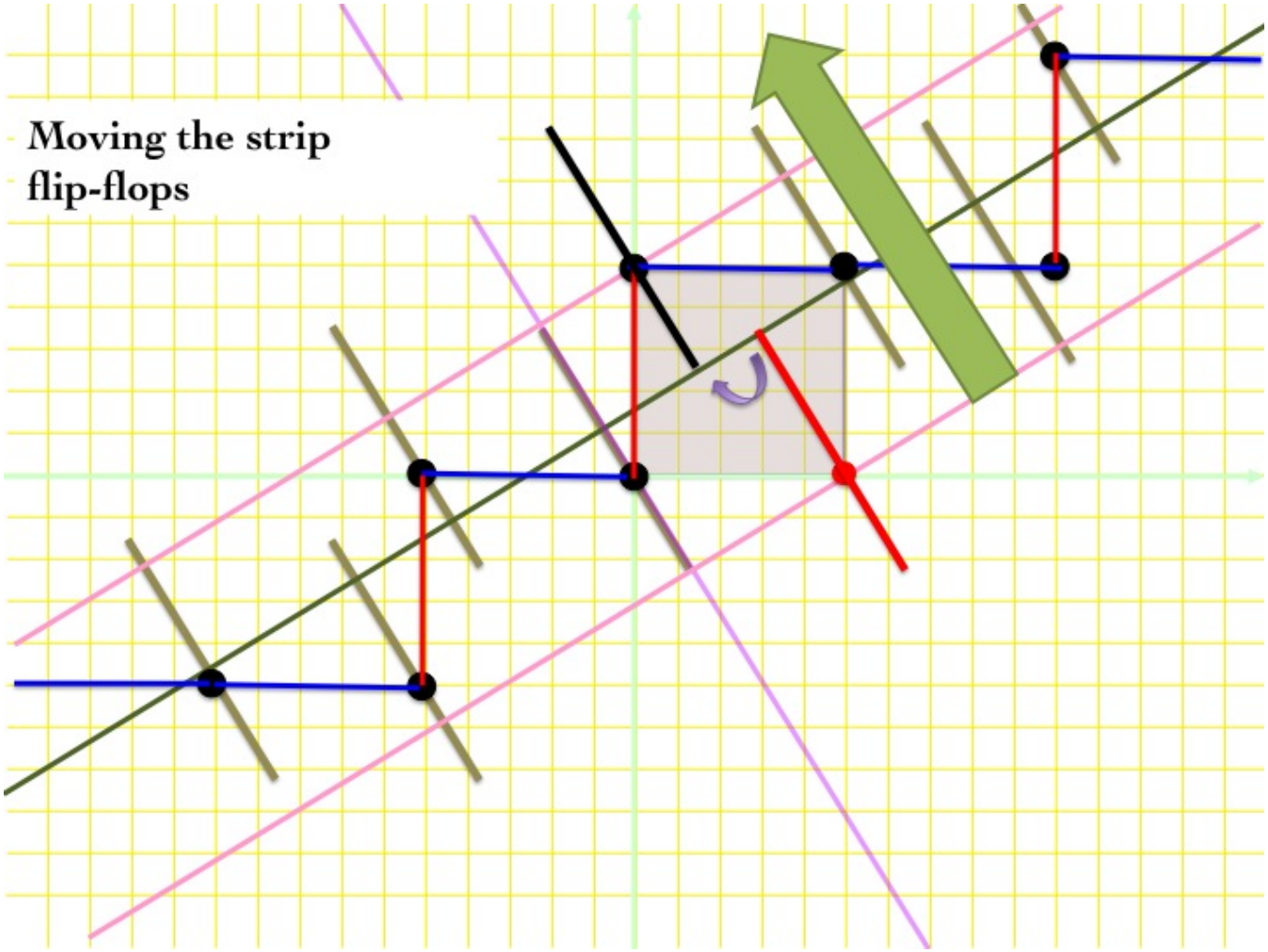

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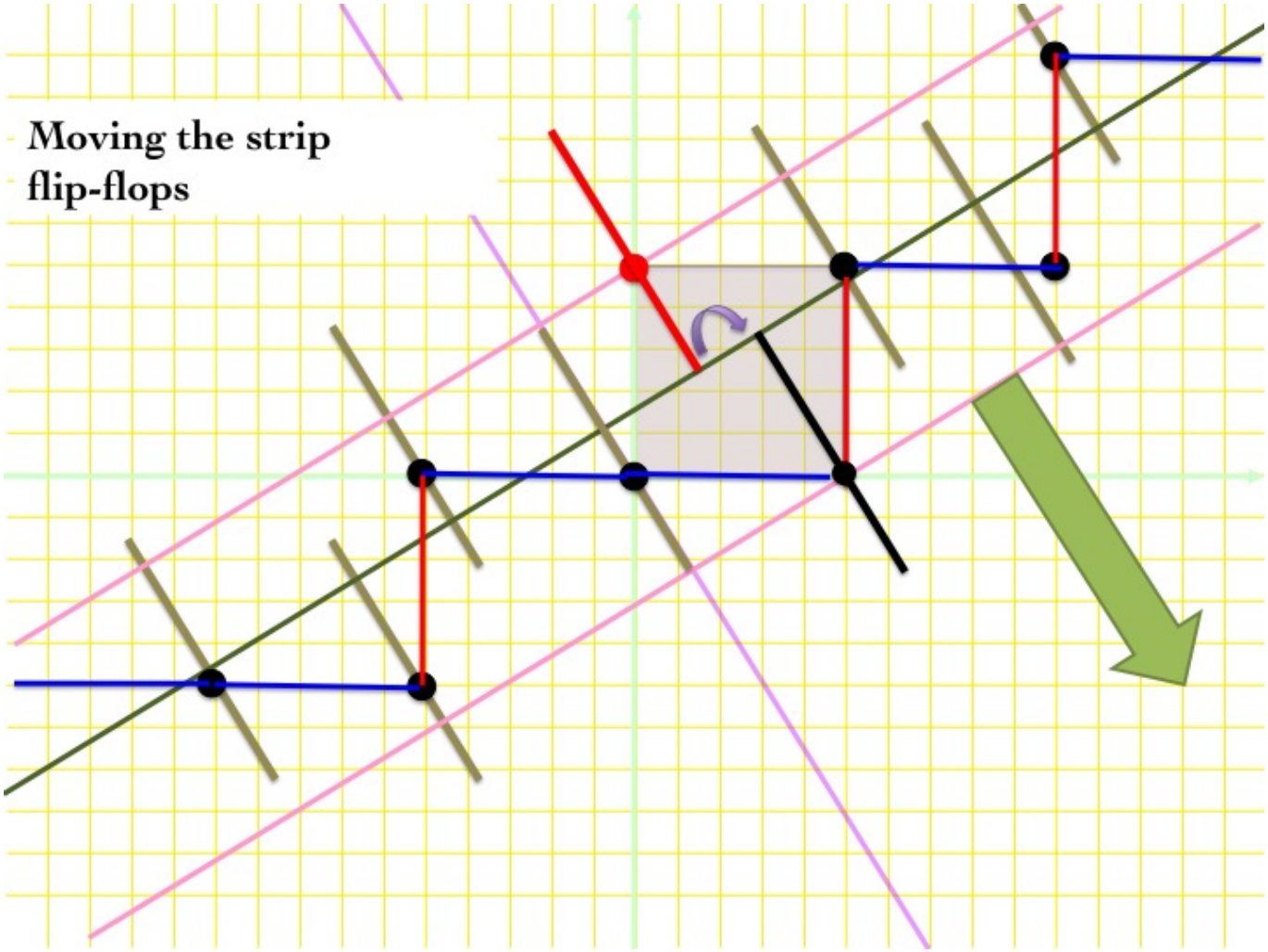

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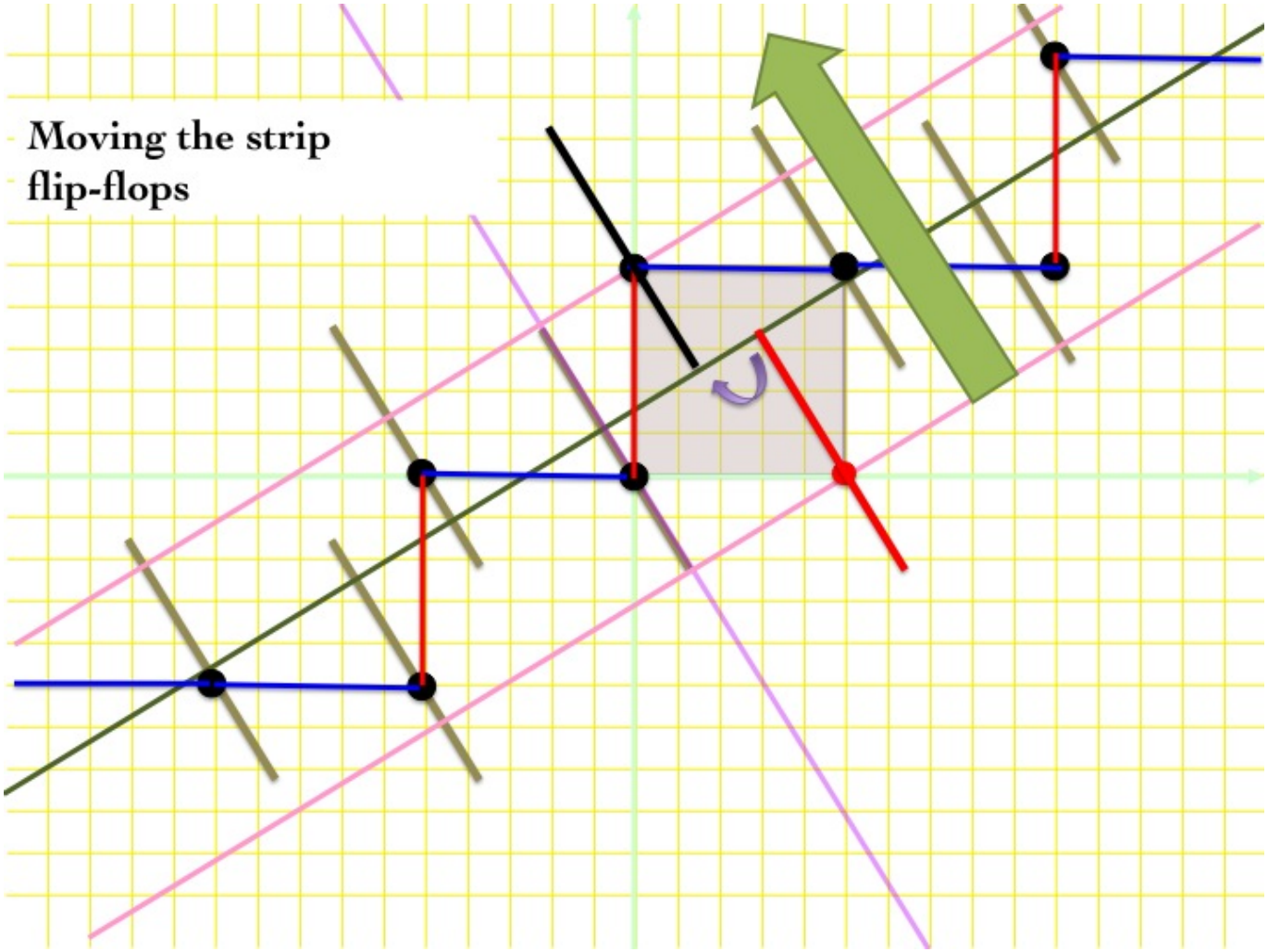

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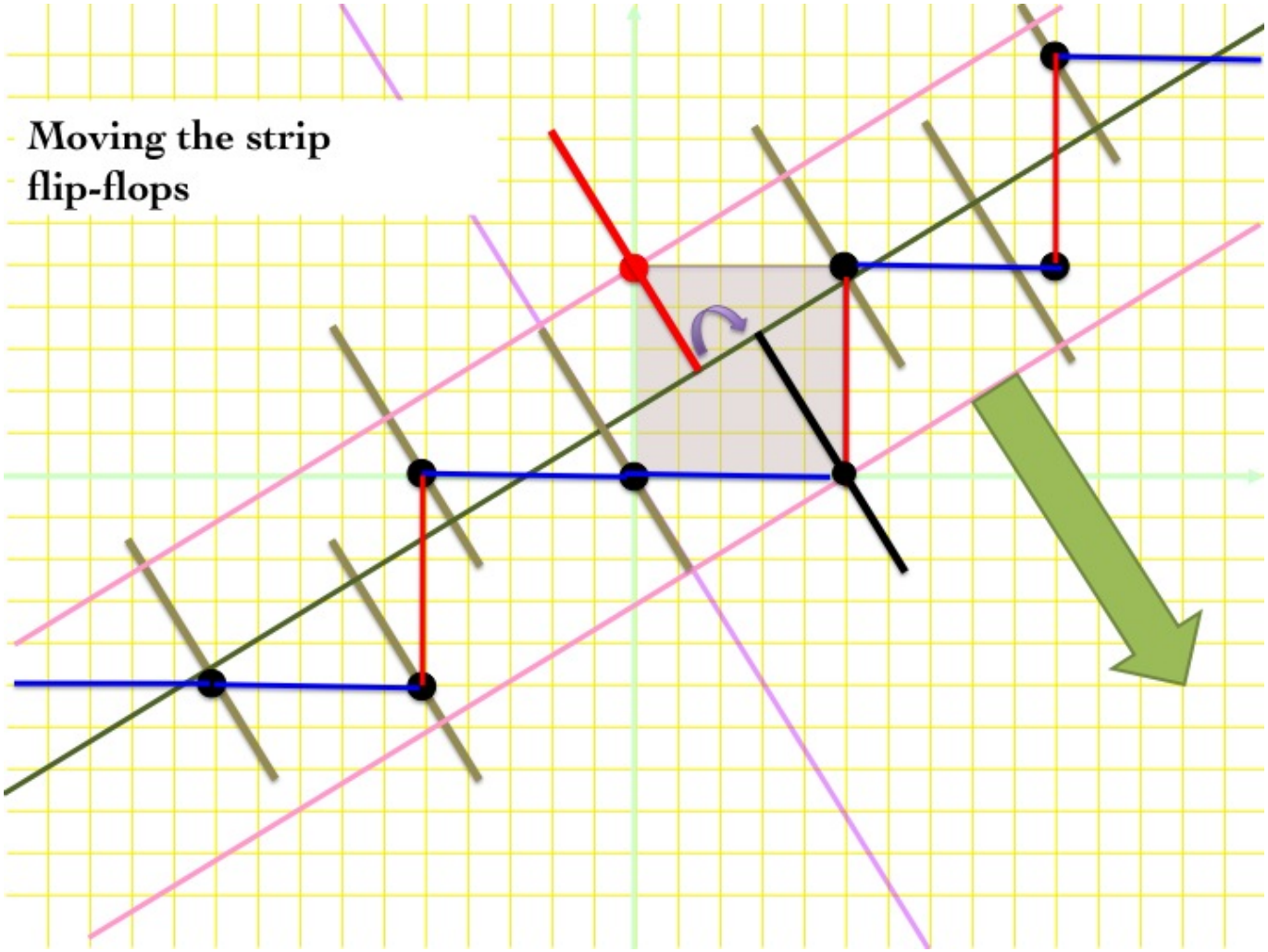

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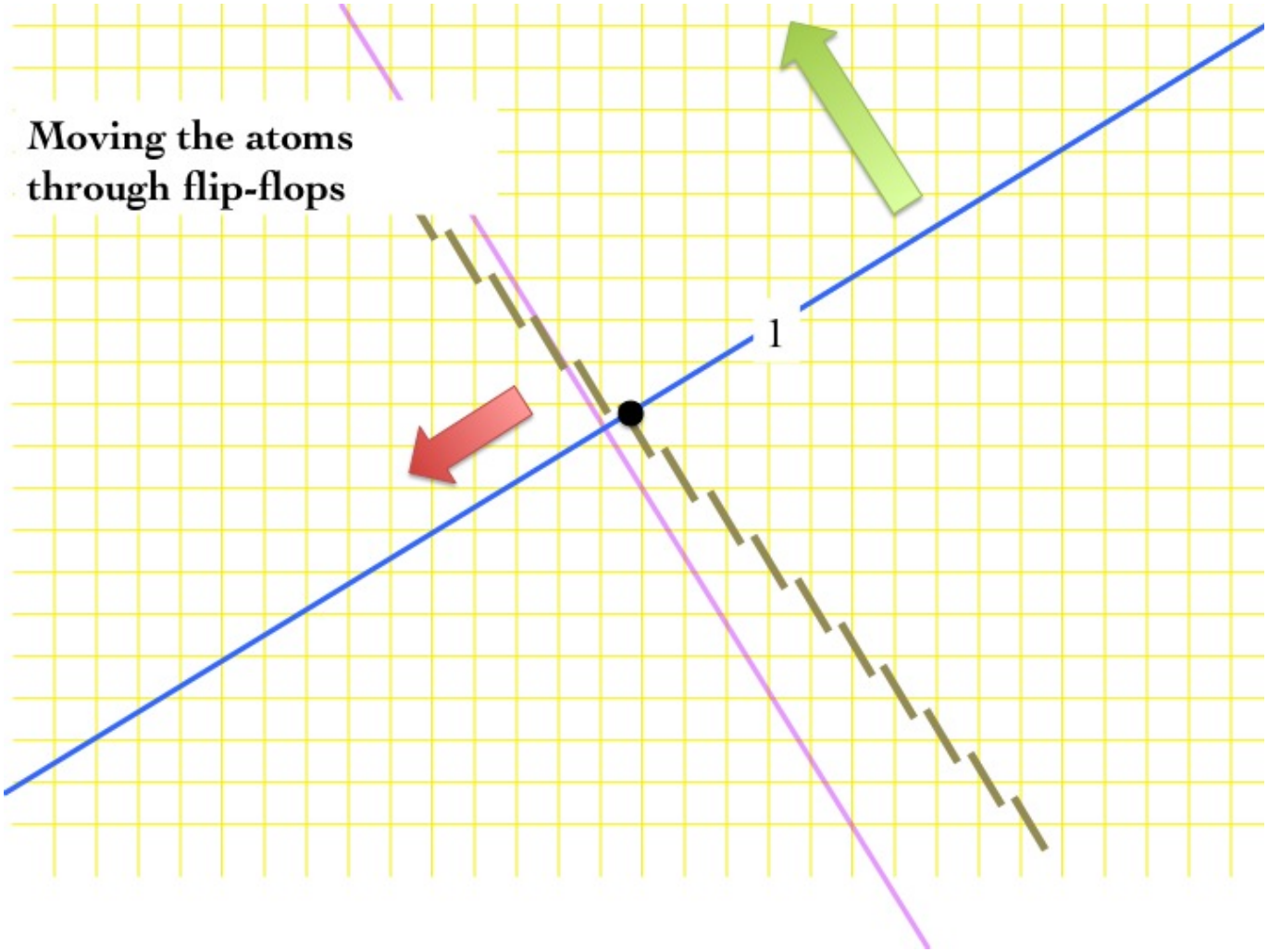

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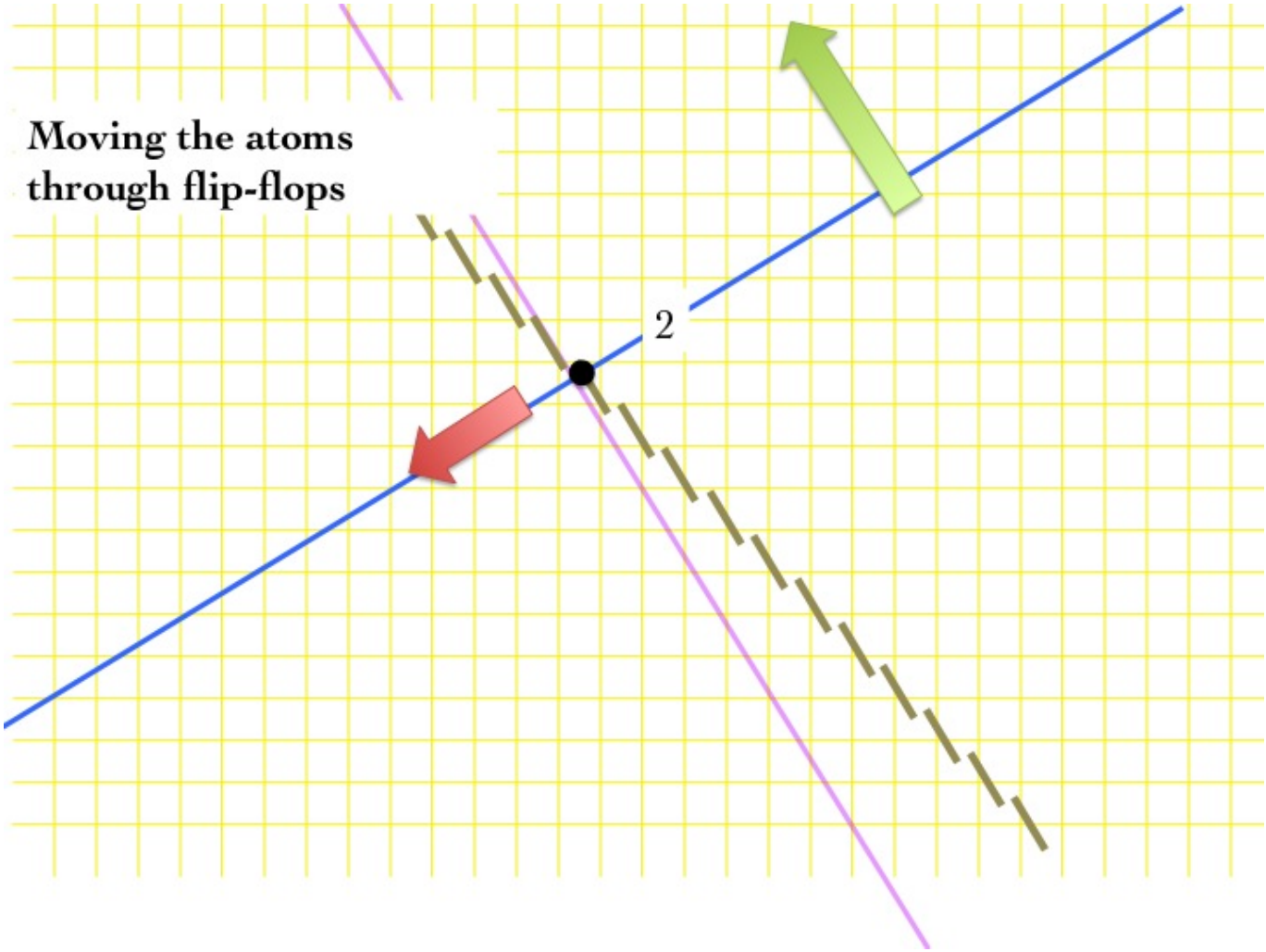

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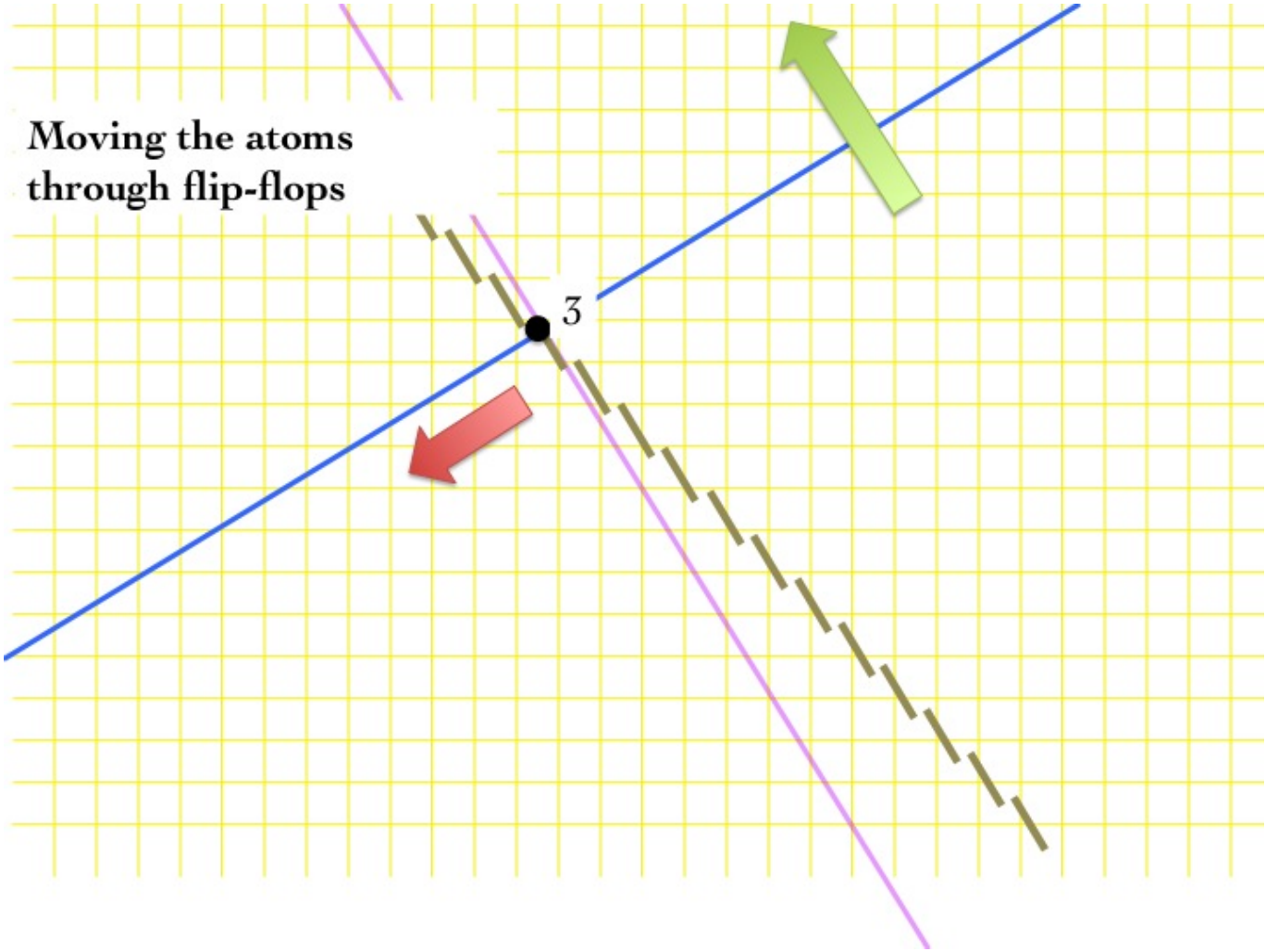

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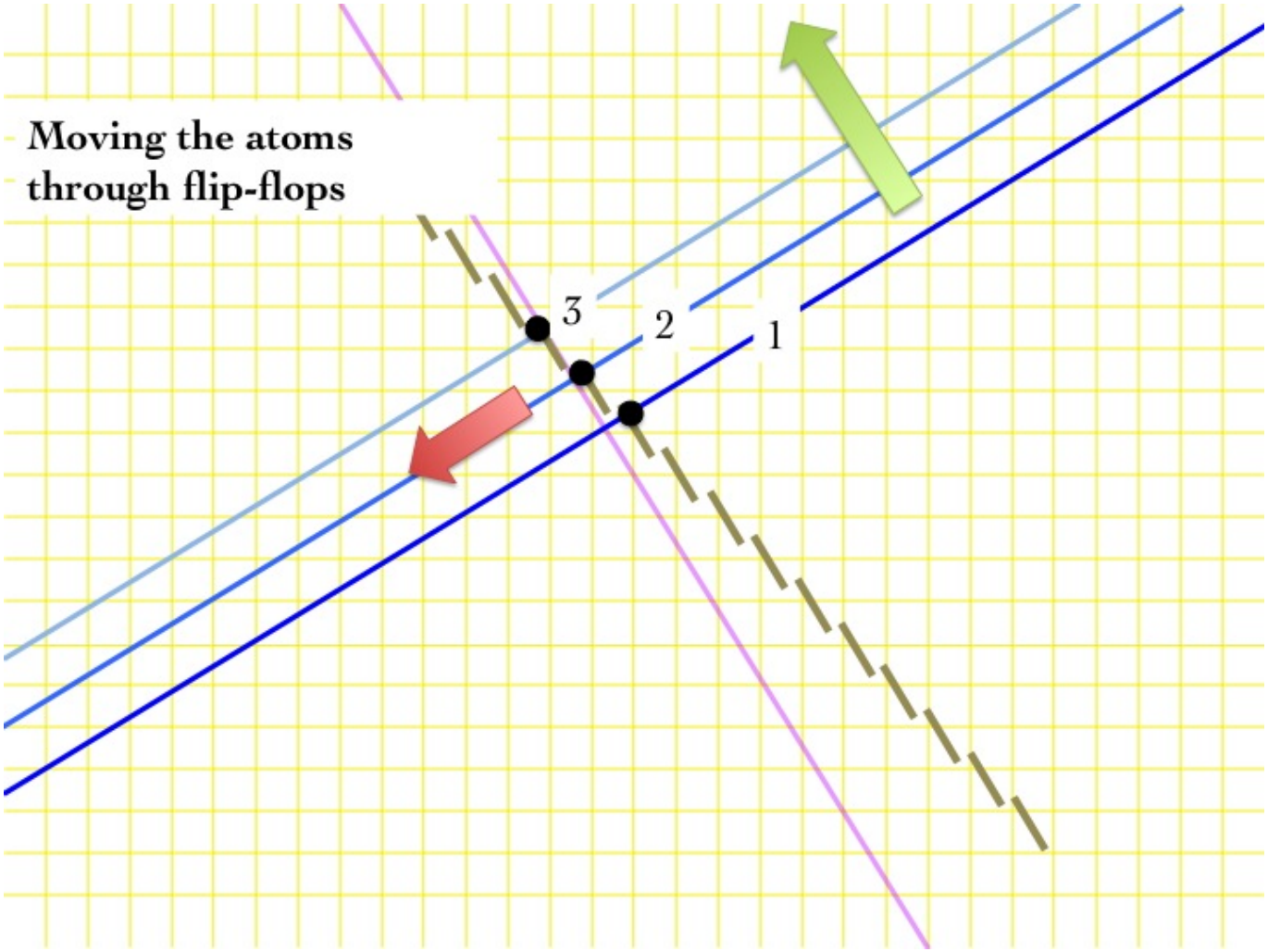
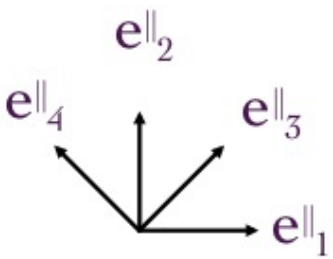
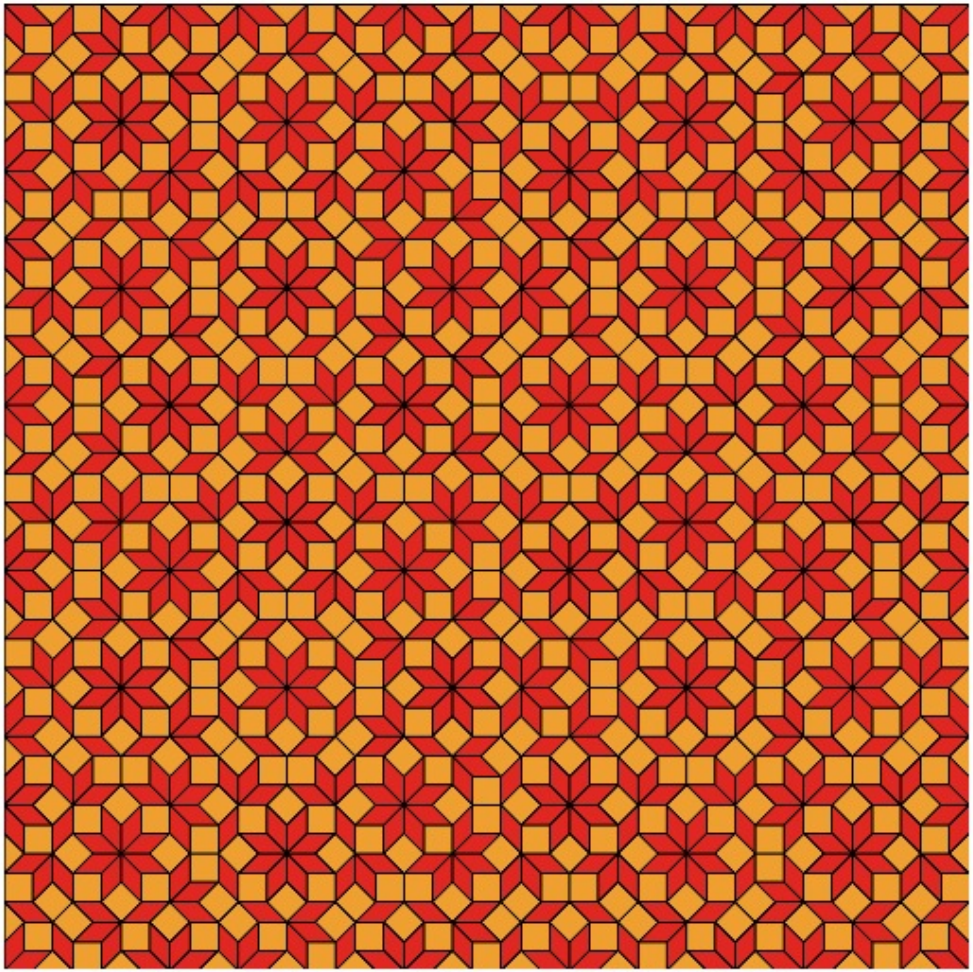


The Octagonal Tiling

Octagonal Lattice

$Z^4 \rightarrow R^2$





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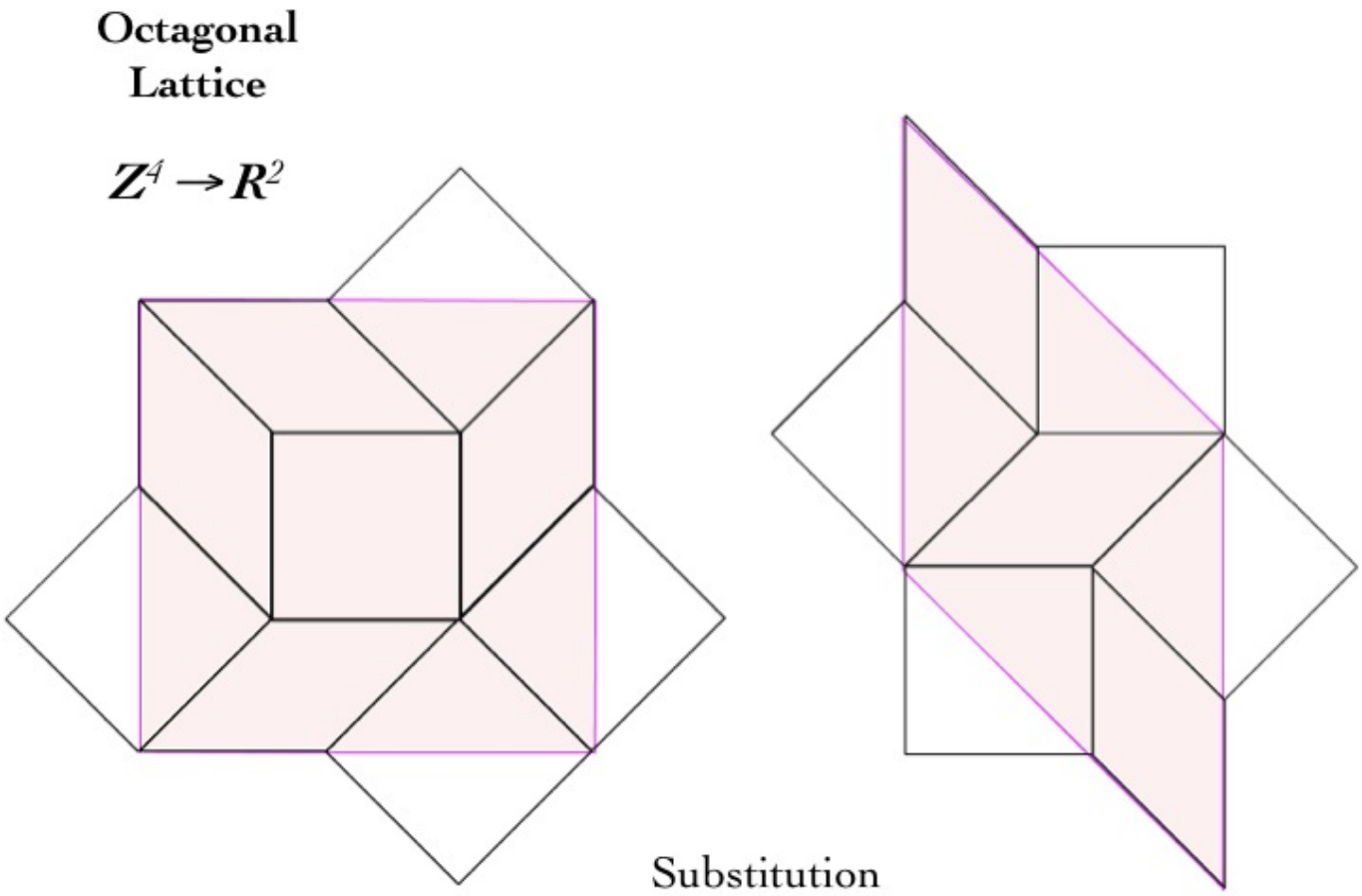

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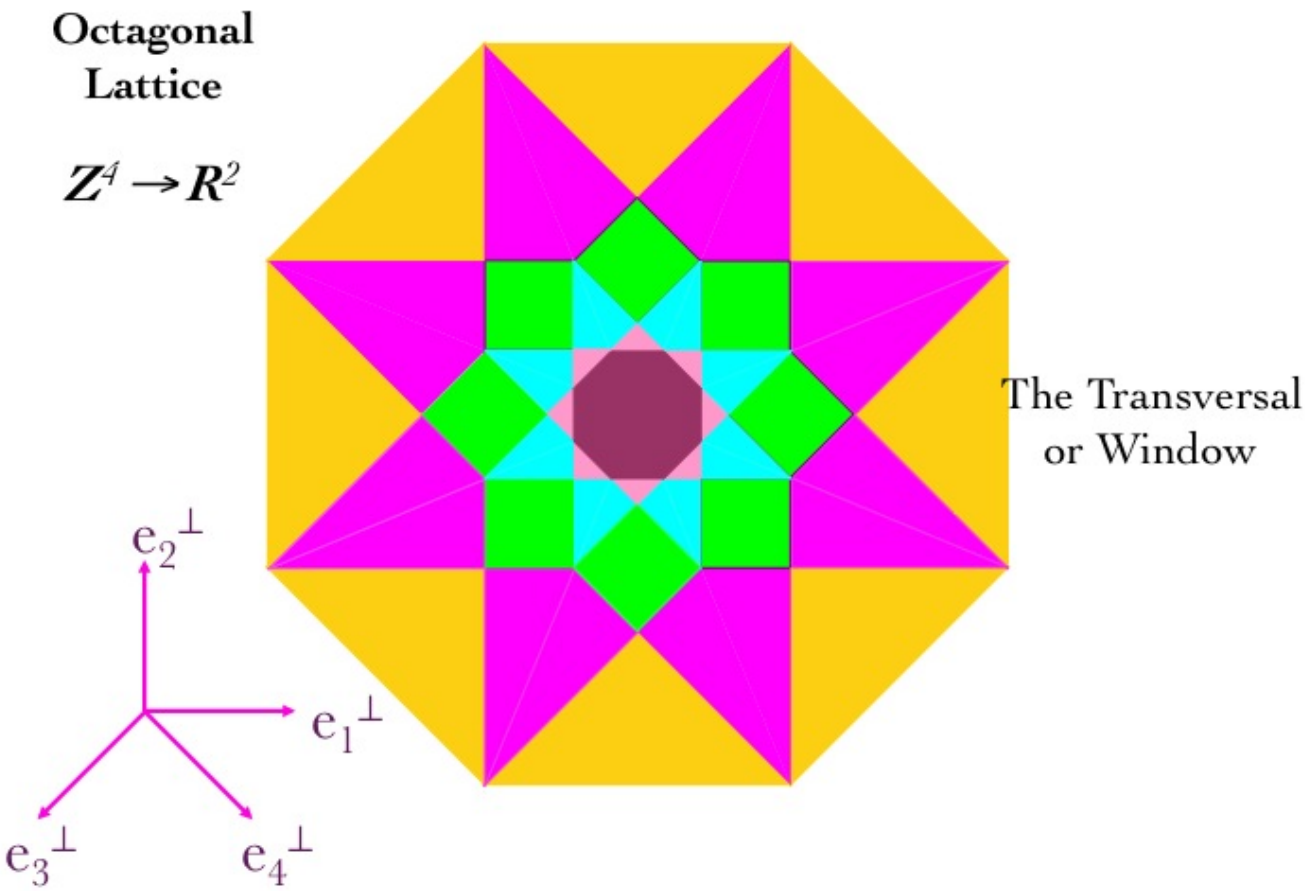

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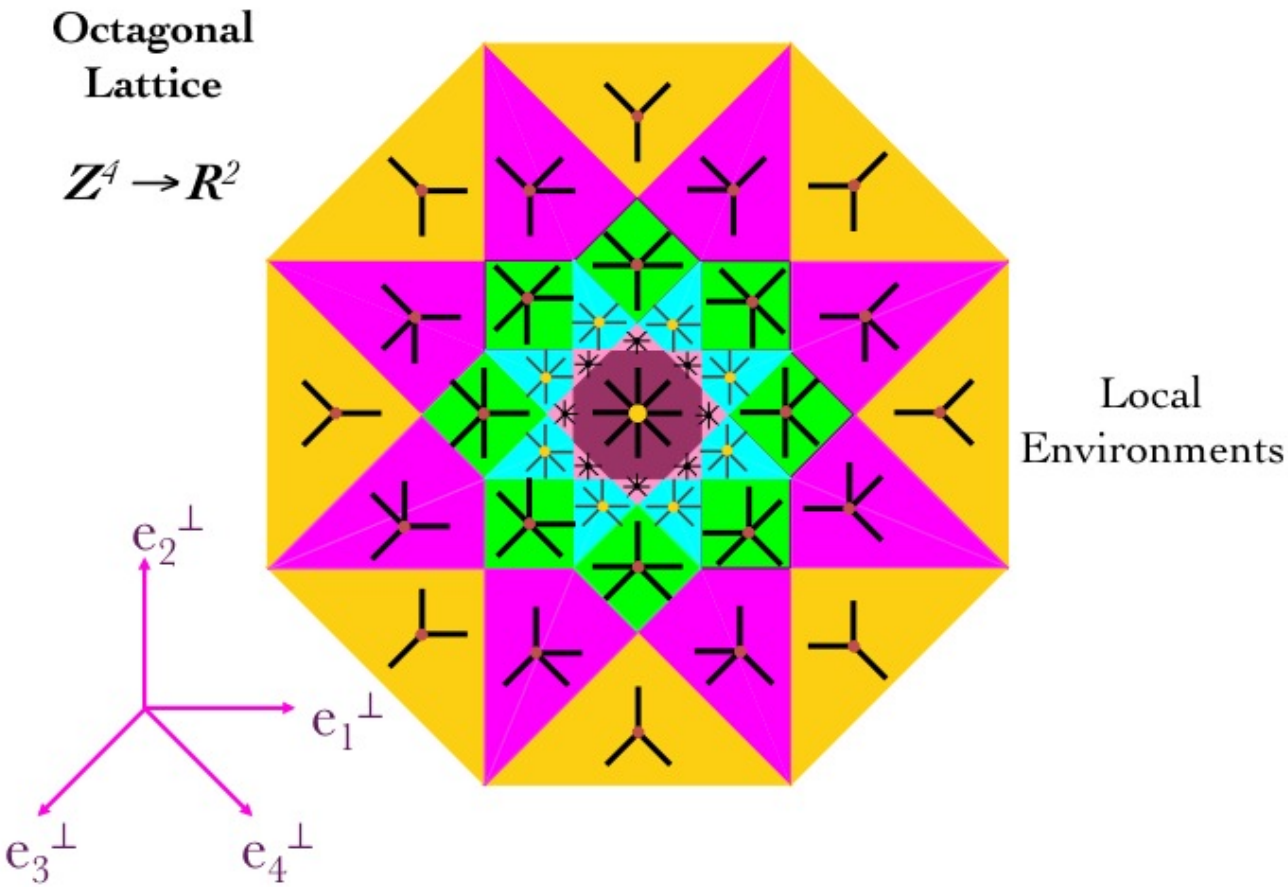

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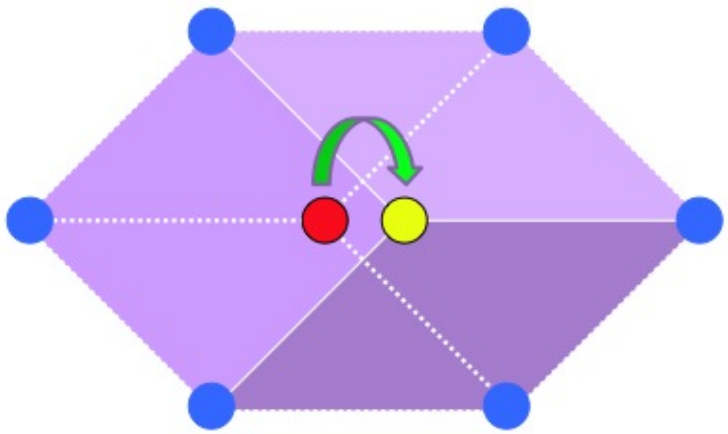

Flip-Flops

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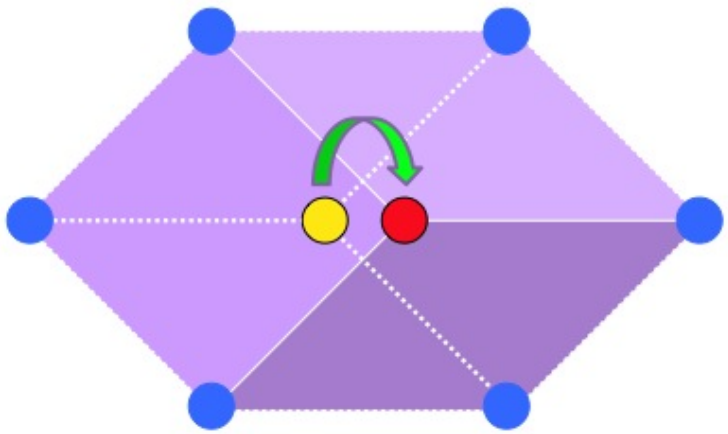

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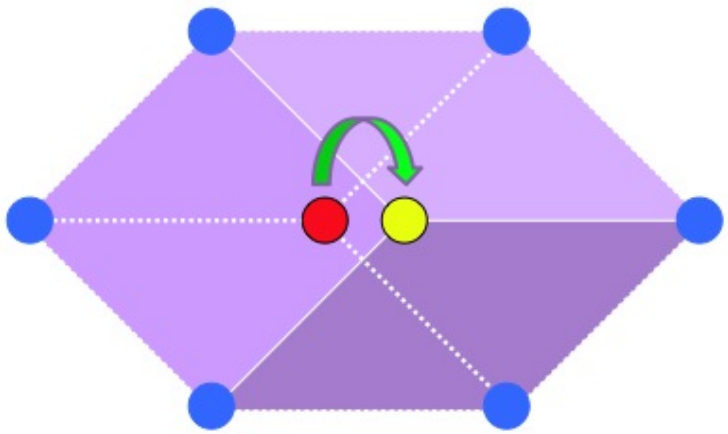

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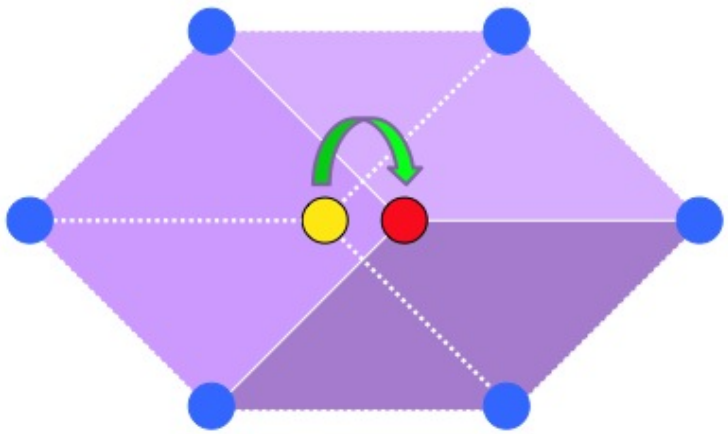

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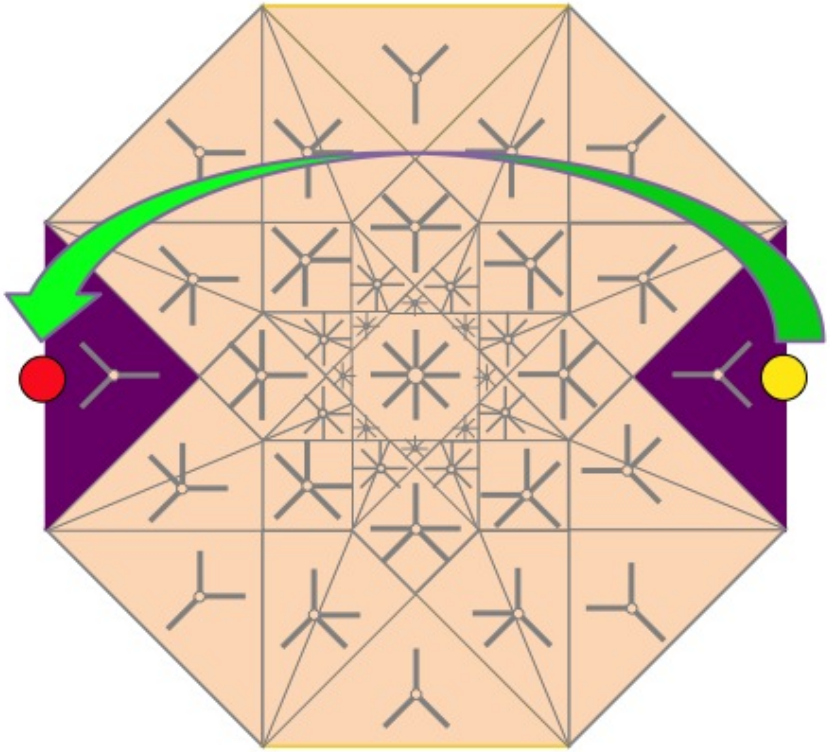

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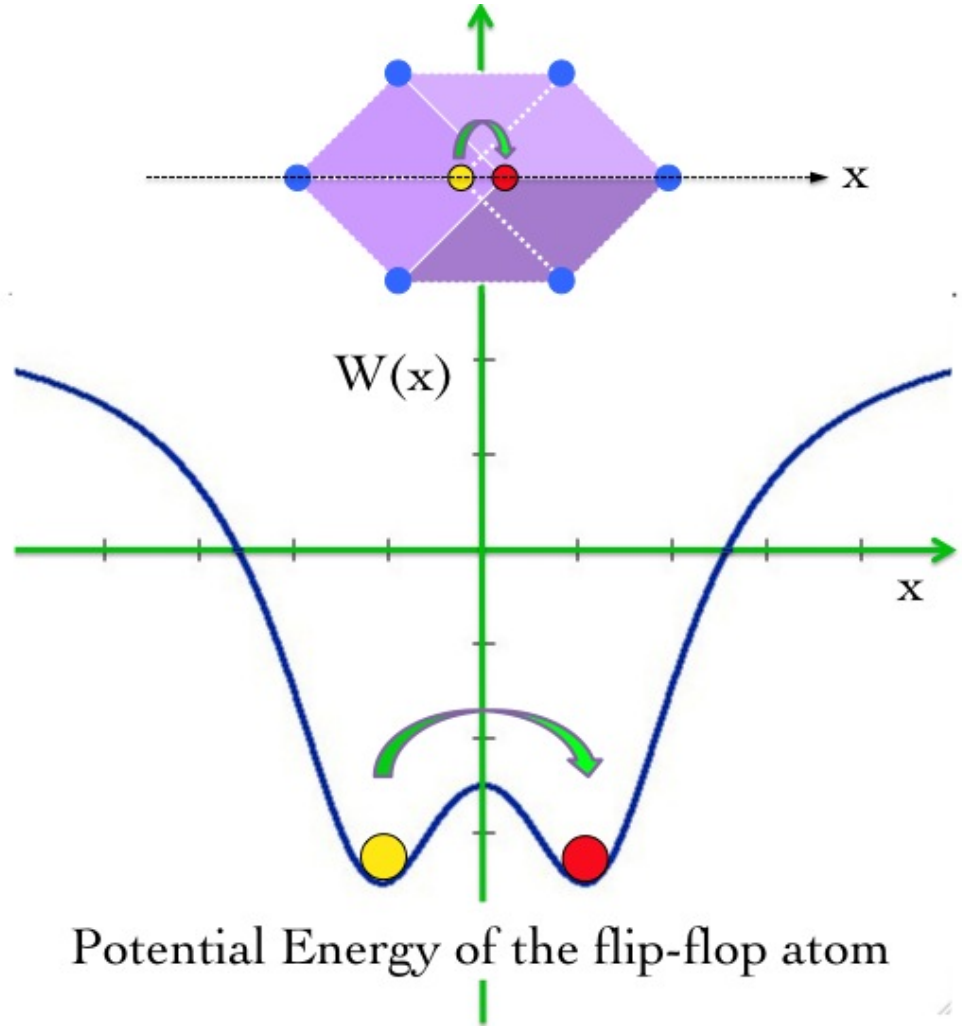
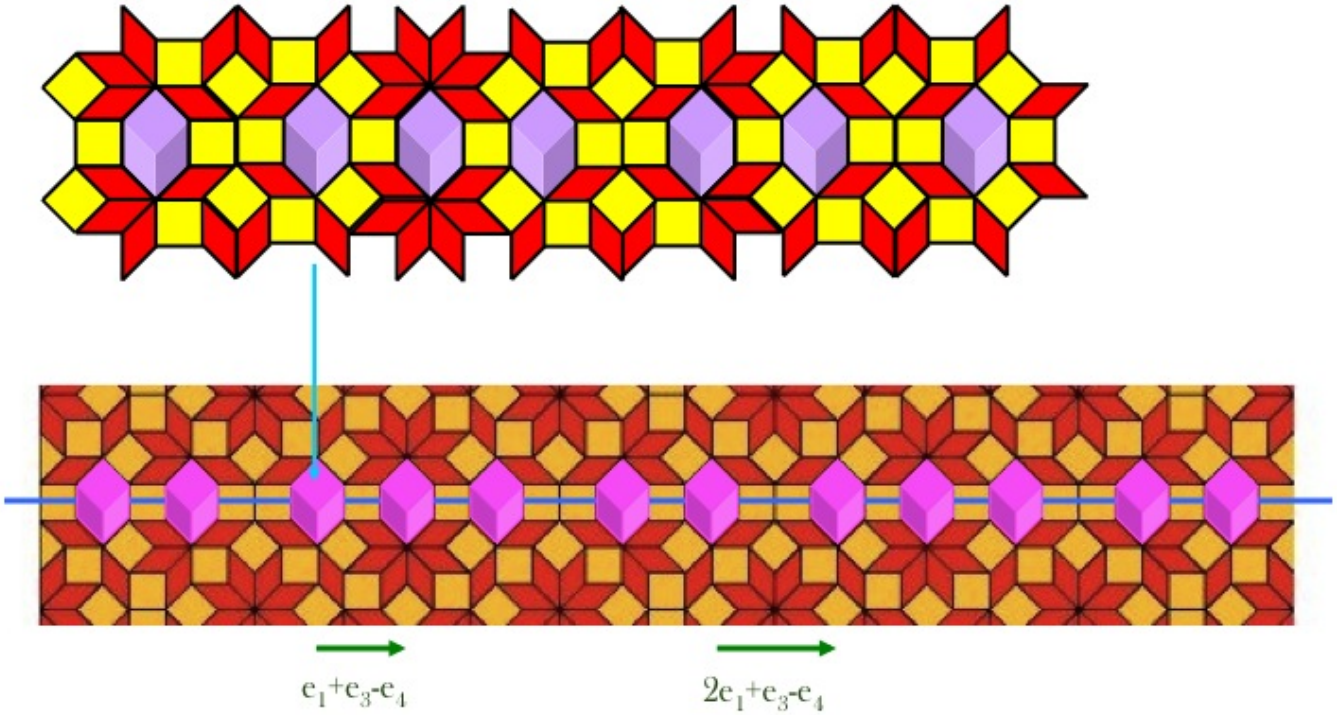


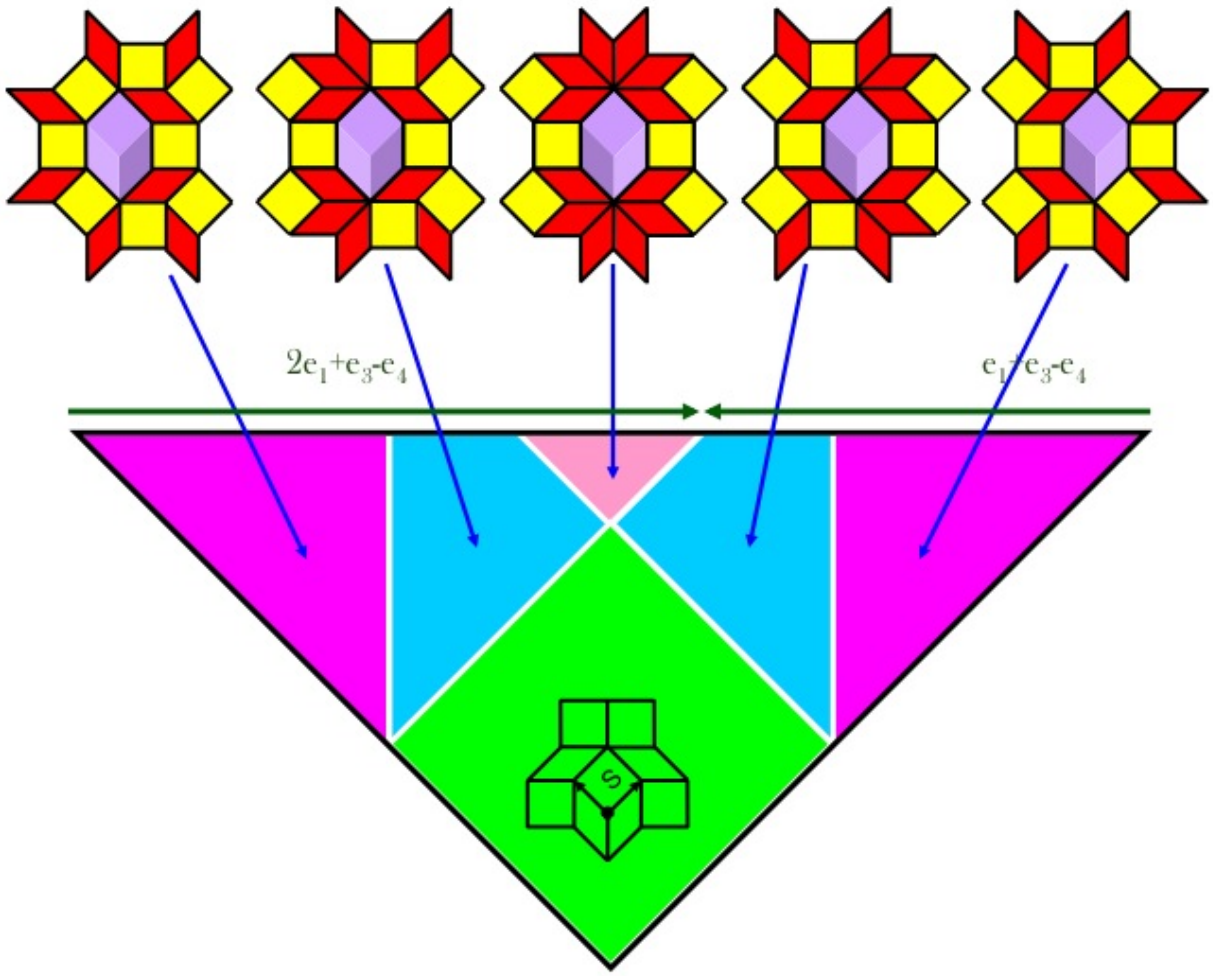
Potential Energy of the flip-flop atom

The Octagonal Tiling



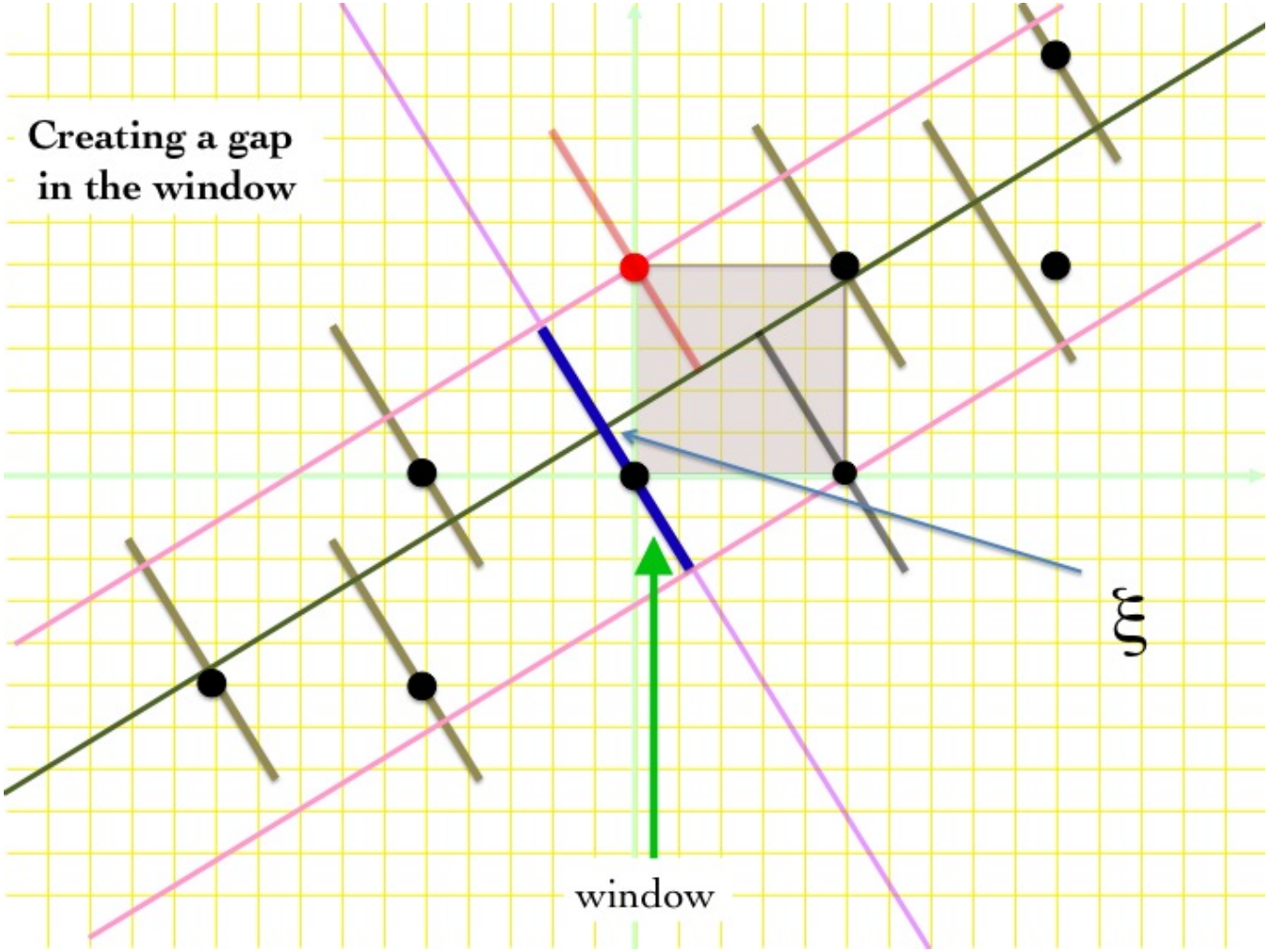
A line of flip-flops with a symmetric environment

The Octagonal Tiling



II - Transversal

Cantor Structure



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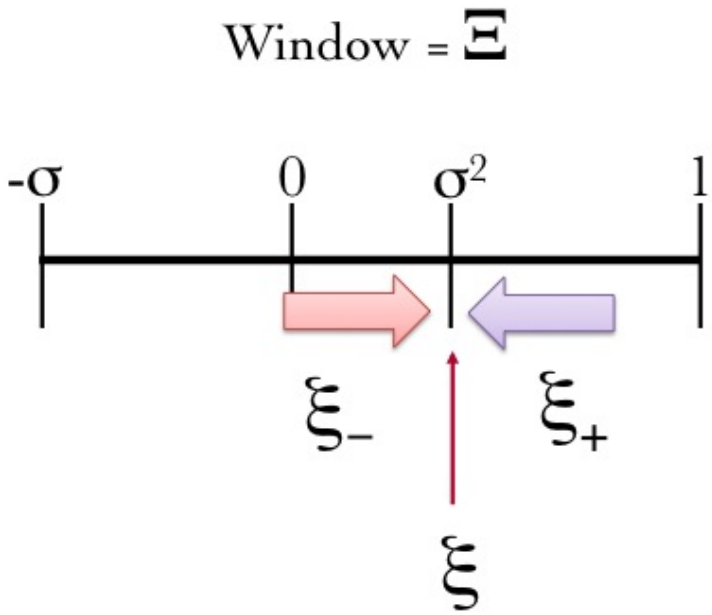

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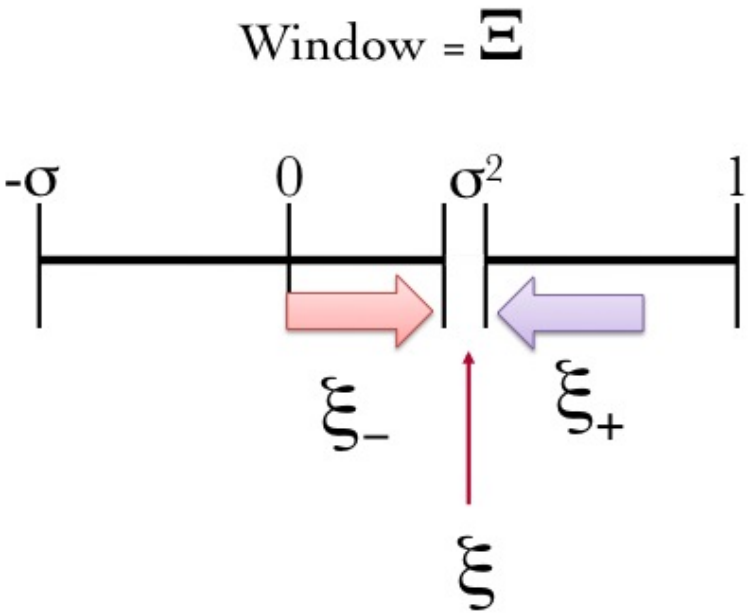

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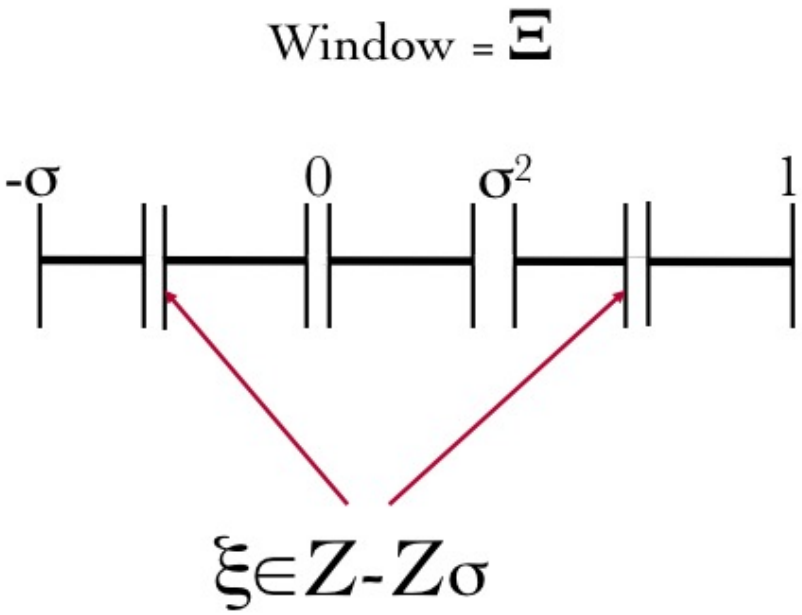

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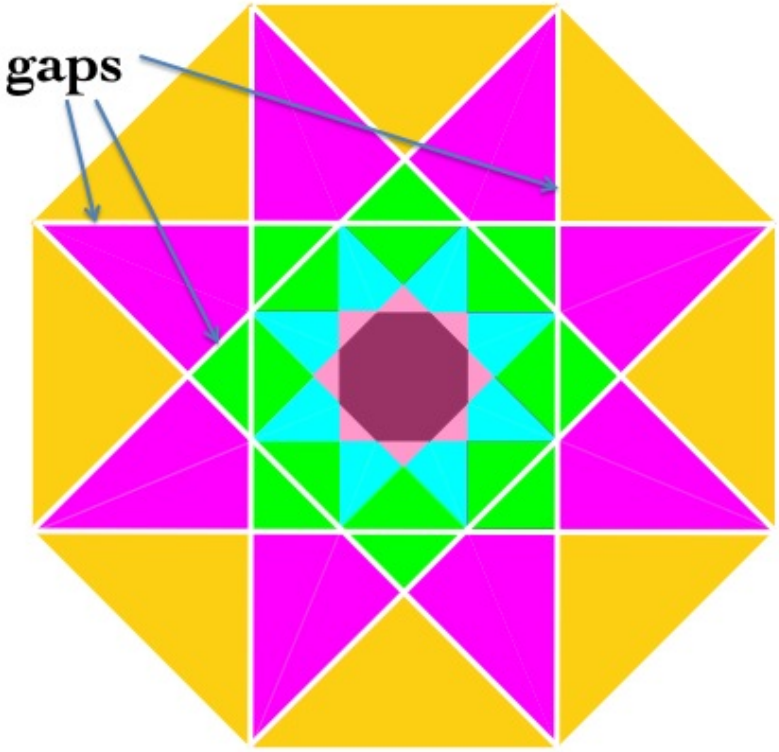
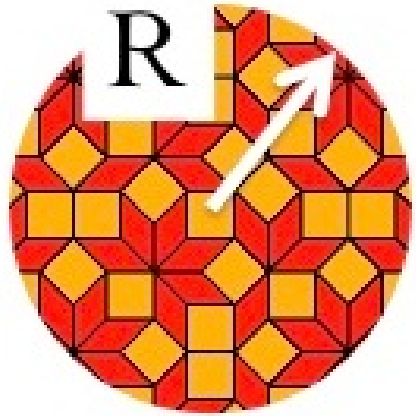


Inverse Limit

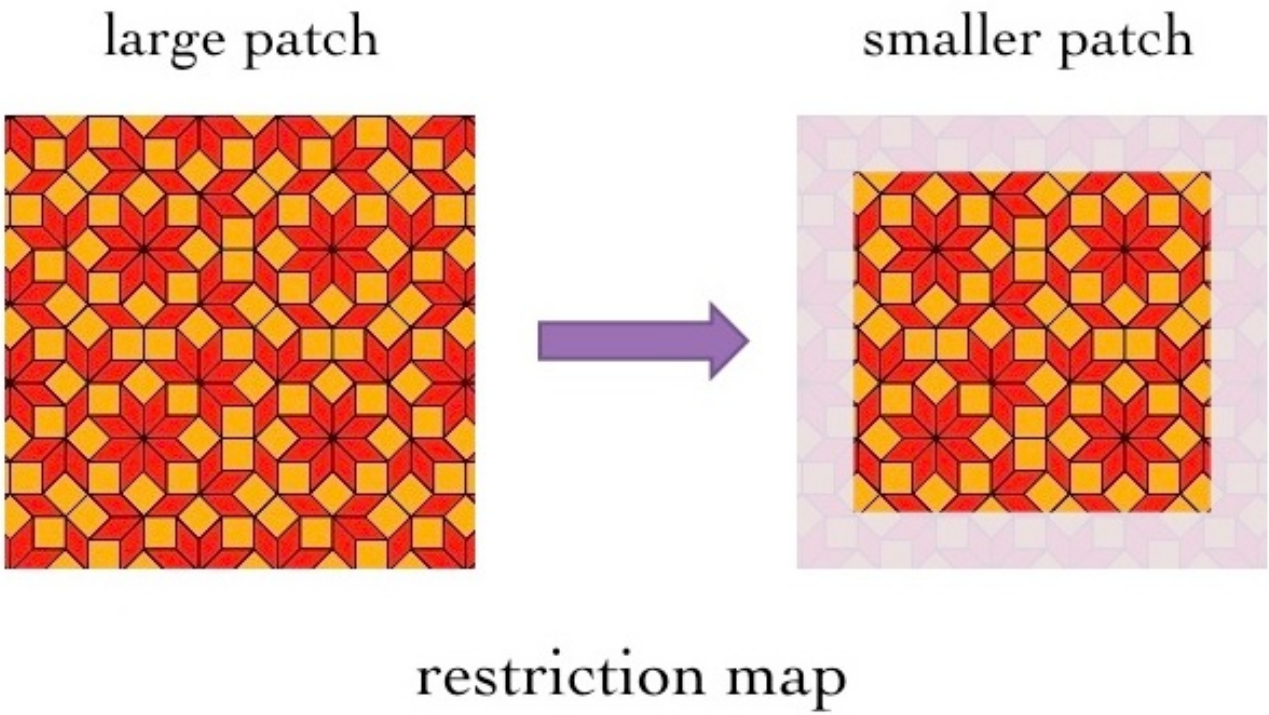
a patch of radius R



Let $\mathscr{P}_R$ be the set of patches of radius R, modulo translation.

The tiling has *finite local complexity* (FLC), if and only if $\mathscr{P}_R$ is a *finite* set for all R. In particular $R \to \mathscr{P}_R$ is *locally constant* and *non-decreasing*. Thus there is a sequence $R_0 = 0 < R_1 < \cdots < R_n < \cdots$ with $R_n \to \infty$ such that $\mathscr{P}_R = \mathscr{P}_n$ for $R_n \leq R < R_{n+1}$.

Inverse Limit



There is a restriction map $\pi : \mathscr{P}_{n+1} \to \mathscr{P}_n$. Then the *transversal* is defined by the inverse limit

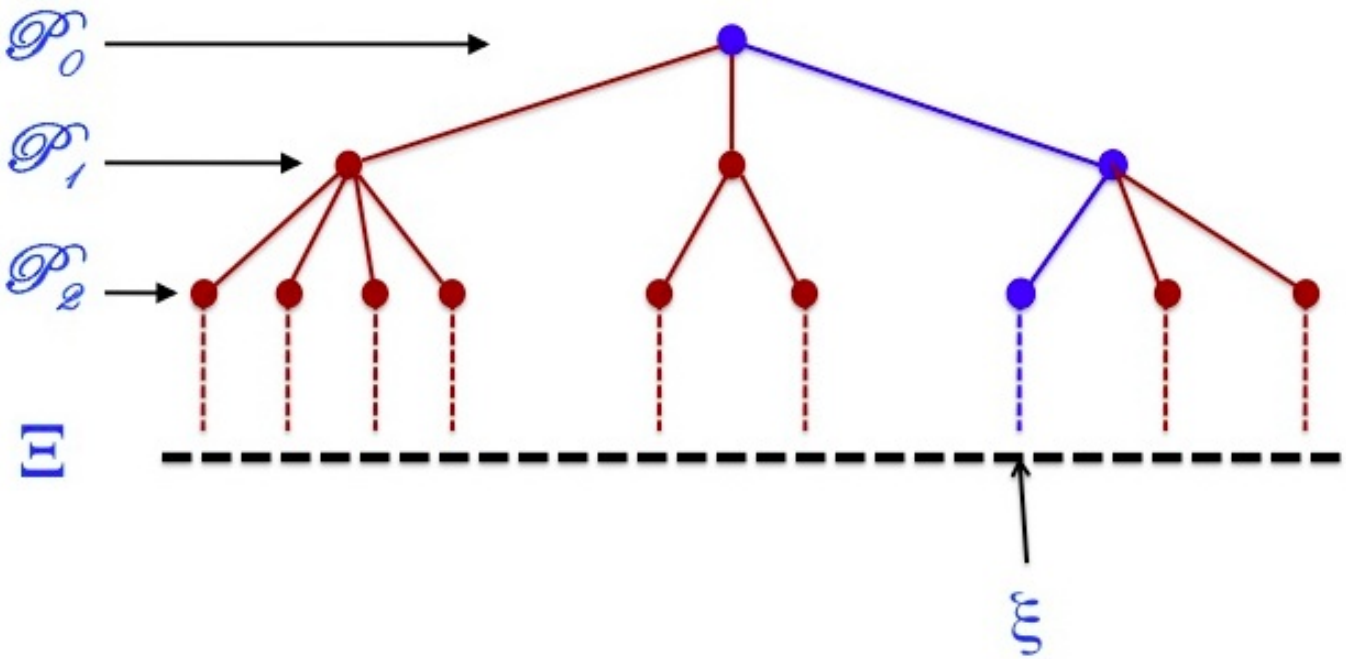
$$\Xi = \lim_{\leftarrow \pi} \mathscr{P}_n$$

Rooted Tree

Since all the $\mathscr{P}_n$'s are finite set, Ξ is a *Cantor set*.

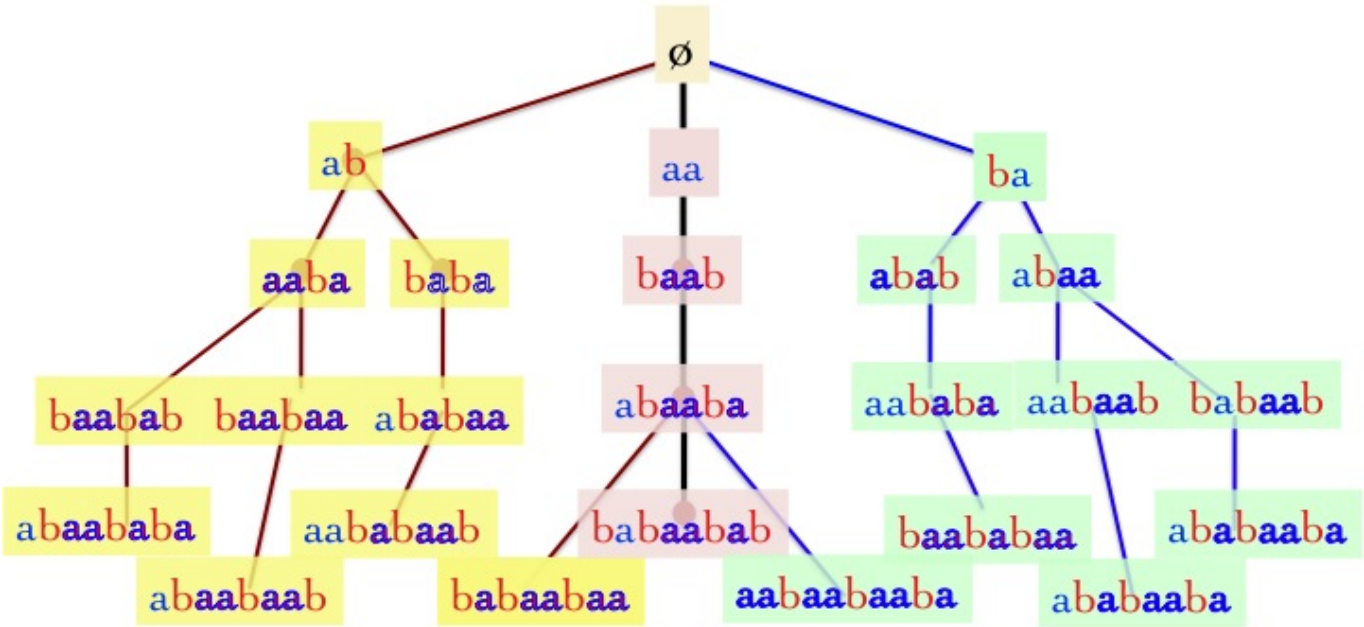
A point of Ξ is an infinite sequence $\xi = (p_n)_{n=0}^{\infty}$ of compatible patches, so it defines a unique *tiling*.

This inverse limit can be represented by a *rooted tree*



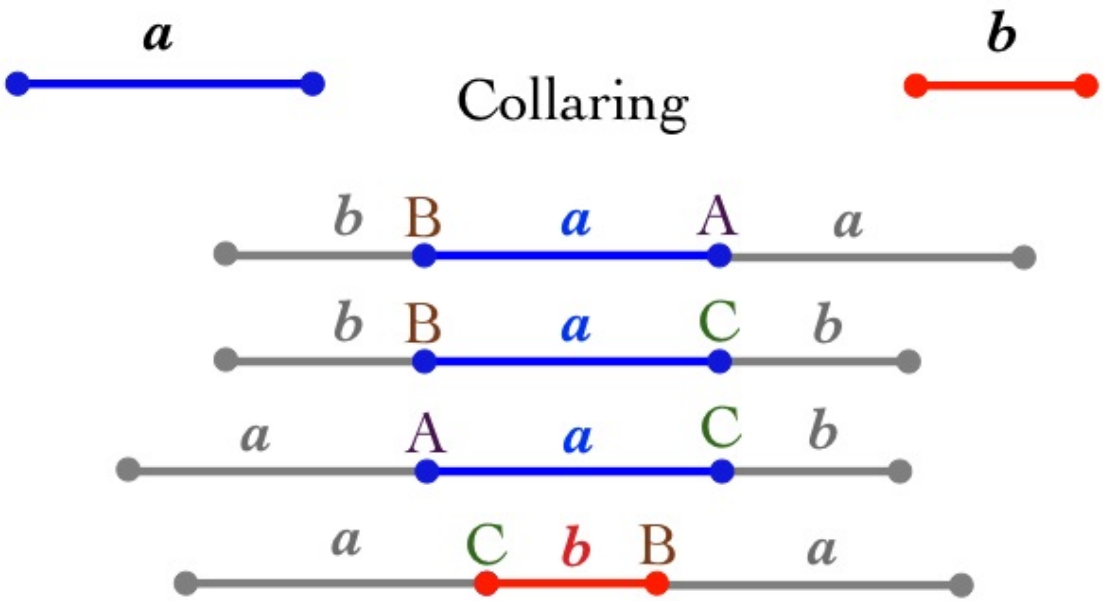
Rooted Tree

For the *Fibonacci sequence* this gives



The Fibonacci Tree

Bratteli Diagram



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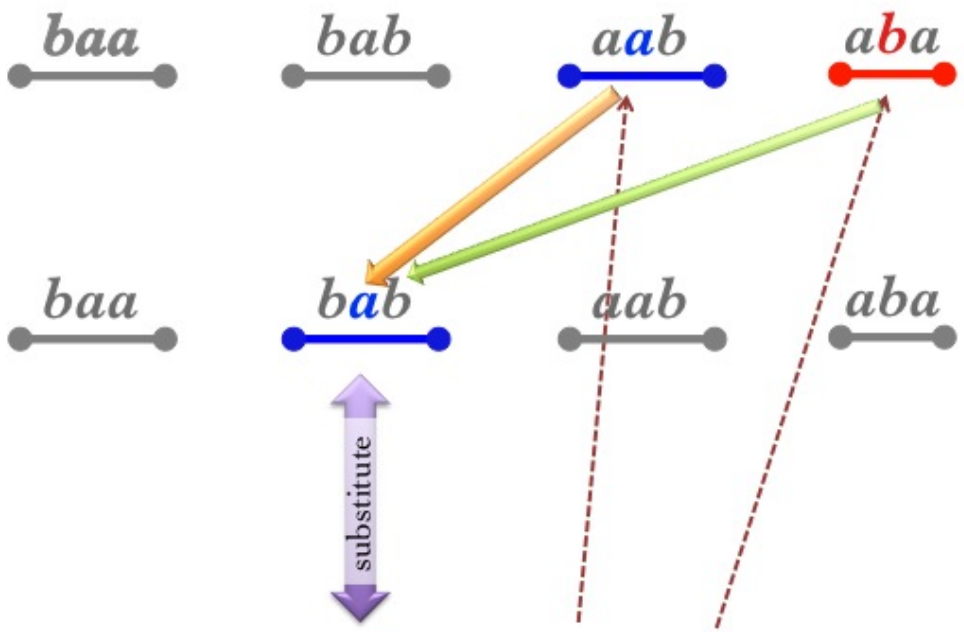
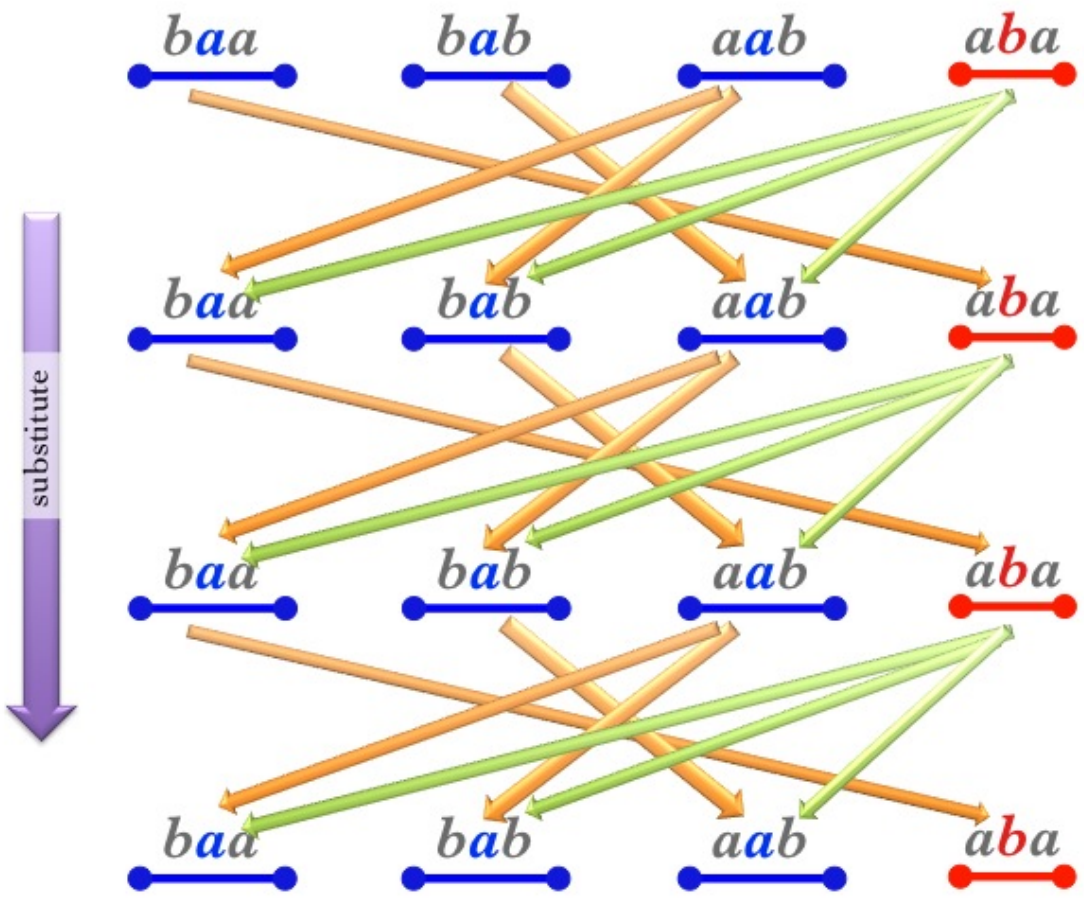


$aaba = (aab)(aba)$

positions = 0 (left), 1 (right)

Bratteli Diagram



Bratteli Diagram

An *infinite path* of the *Bratteli diagram* gives a tiling of the transversal, where the origin is uniquely defined.

Conversely, any tiling in Ξ gives rise in a unique way to an infinite path in the Bratteli diagram.

If $\mathcal{B}$ denotes the Bratteli diagram, and if $\partial\mathcal{B}$ denotes its set of infinite path starting at the top, then there is an homeomorphism

$$\phi : \partial\mathcal{B} \to \Xi$$

III - Spectral Triples

Spectral Triples

A *spectral triple* for a C*-algebra $\mathcal{A}$ is a family $X = (\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{H}$ is a Hilbert space, D and unbounded operator on $\mathcal{H}$ such that

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Proposition: *Then $C^1(X)$ is a dense $*$-subalgebra of $\mathcal{A}$, invariant under the holomorphic functional calculus.*

Example

Let M be a *spin^c Riemannian manifold*, $\mathcal{A} = C(M)$, $\mathcal{H}$ the space of L^2-sections of the *spin bundle* and D the corresponding *Dirac operator*, where $\mathcal{A}$ acts by pointwise multiplication.

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Theorem (Connes) *The family $X_M = (\mathcal{A}, \mathcal{H}, D)$ above is a spectral triple. The geodesic distance between $x, y \in M$ can be recovered through*

Example

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Hence the algebra $\mathcal{A}$ encodes the *space*, the Dirac operator D encodes the *metric*. $\mathcal{H}$ is needed to define D.

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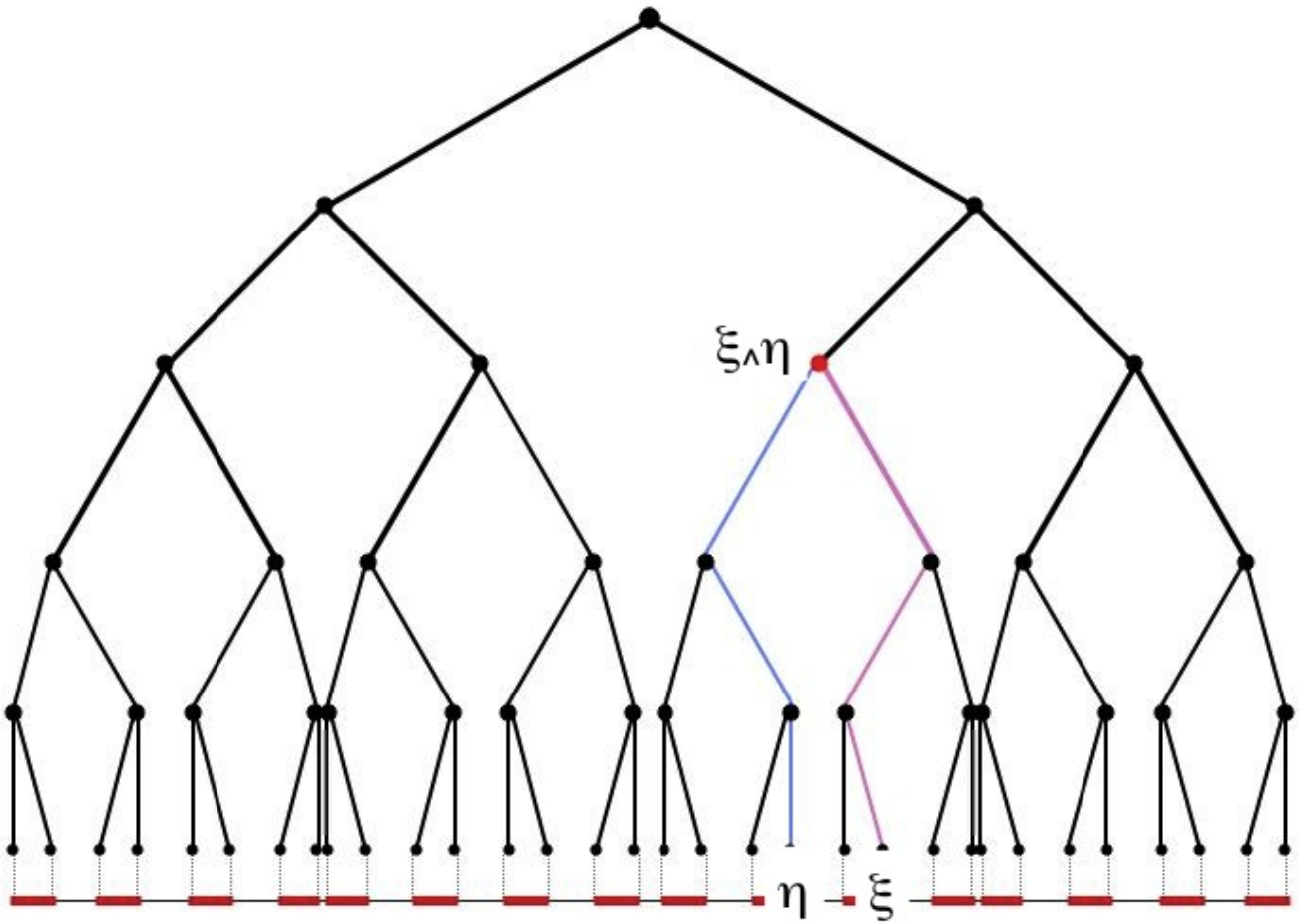
Each ultrametric on Ξ can be obtained in such a way through a rooted tree defined from the metric.

Ultrametric on Ξ



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The Pearson-Palmer Spectral Triple

Given p a patch, let $\Xi(p)$ be the set of all tilings in Ξ compatible with p at the origin. The family $(\Xi(p))_{p\in\mathscr{P}}$ is a basis of clopen set for the topology of Ξ.

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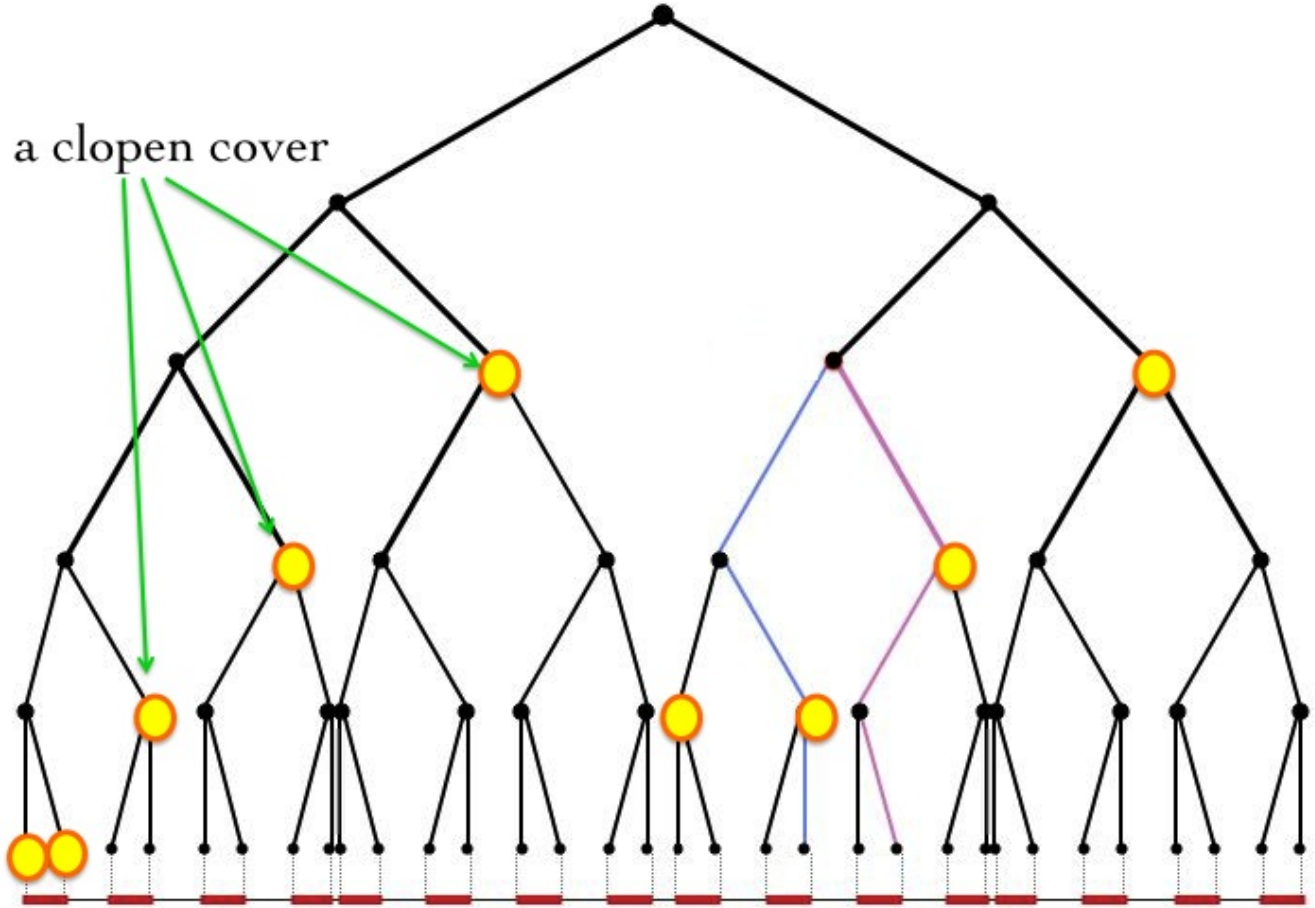

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$$\operatorname{diam}\mathcal{P} = \max\{\kappa(p)\,;\, p \in \mathcal{P}\}$$

An infinite sequence $(\mathcal{P}_n)_{n\in\mathbb{N}}$ of clopen cover is called *resolving* if $\lim_{n\to\infty}\operatorname{diam}\mathcal{P}_n = 0$.

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- **Representation:** for each choice τ and $f \in C(\Xi)$

$$(\pi_\tau(f)\psi)(p) = \begin{bmatrix} f(\xi_p) & 0 \\ 0 & f(\eta_p) \end{bmatrix} \psi(p)\,.$$

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There exists a (non unique) resolving sequence of clopen covers $(\mathcal{P}_n)_{n\in\mathbb{N}}$, called a Hausdorff sequence, such that $s_0 = \dim_H(\Xi)$.

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The Connes state is defined by

$$\mathcal{T}(f) = \lim_{s \to s_0} \frac{1}{\zeta(s)} \mathrm{Tr}\left(\frac{1}{|D|^s}\, \pi_\tau(f)\right), \qquad f \in \mathcal{C}(\Xi)$$

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If so, $\mathcal{T}$ coincides with the normalized Hausdorff measure on Ξ.

IV - The Pearson Laplacian

Directional Derivative, Tangent Space

If $\tau(p) = (\xi_p, \eta_p)$ then

$$[D, \pi_\tau(f)]\ \psi\ (p) = \frac{f(\xi_p) - f(\eta_p)}{d(\xi_p, \eta_p)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \psi(p)$$

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- $[D, \pi_\tau(f)]$ could be written as $\nabla_\tau f$.

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Hence ν_p can be interpreted as the average over the tangent unit sphere at p.

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The Pearson quadratic form is defined by (if $f, g \in C(\Xi)$)

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The eigenspaces are common to all s's and can be explicitly computed.

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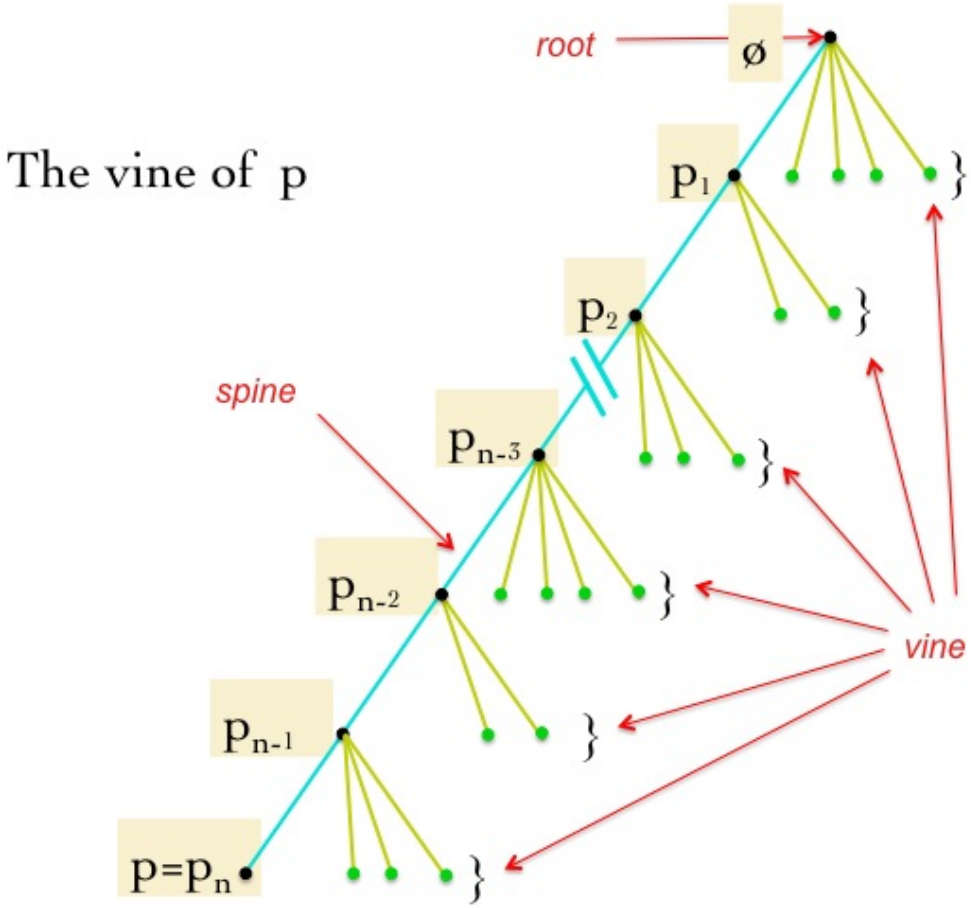

The vine of a vertex v

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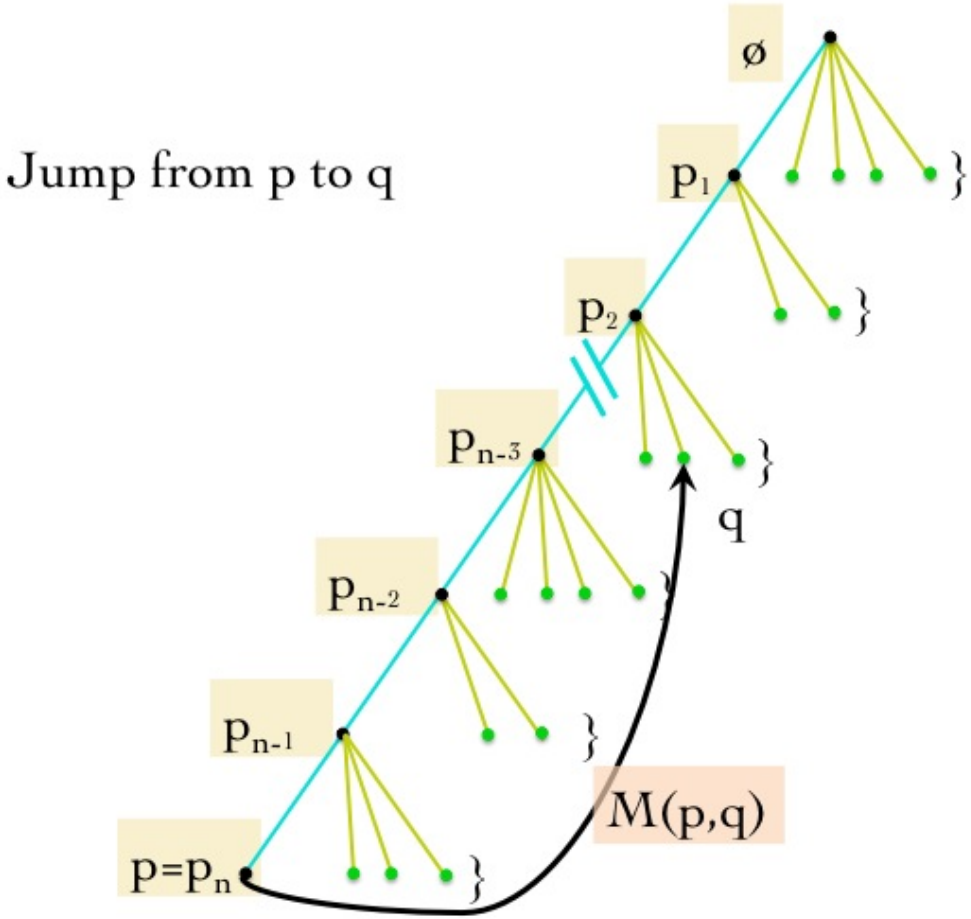
where $M(p,q) > 0$ represents the
probability rate (per unit time) for X_t to jump from $\Xi(p)$ to $\Xi(q)$.



Jump process from v to w

Jump Process

Concretely, if $\hat{q}$ denotes the *father* of q (which belongs to the spine)

$$M(p,q) \;=\; 2\kappa(\hat{q})^{s-2}\;\frac{\mu_p}{Z_{\hat{q}}} \qquad\qquad \mu_p = \mu_H(\Xi(p))$$

where $Z_{\hat{q}}$ is the *normalization constant* for the measure $\nu_{\hat{q}}$ on the set of choices at $\hat{q}$, namely

$$Z_{\hat{q}} = \sum_{q' \neq q'' \in \mathrm{Ch}(\hat{q})} \mu_{q'}\,\mu_{q''}$$

where $\mathrm{Ch}(\hat{q})$ denotes the set of children of $\hat{q}$.

V - Open Problems

Jump Process and Flip-Flops

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Problem: *Is it possible to see the limiting dynamics in terms of the Pearson Laplacian ?*

Atomic Motion by Flip-Flops

How can one put the atomic system out of equilibrium to mimic the transverse motion of the space $E_{||}$? In Physics, such motion is triggered by the creation of *generalized forces*.

Problem: *Can one identify these forces ?*

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Problem: *Can one identify these forces ?*

If so, can one write a *transport coefficient* (linear response theory) to compute the current of atoms resulting from applying such a force ? In particular

Problem: *Is there a quantization of this transport coefficient resulting from the topology of the transversal ?*

Fault Lines

Sadun and Frank have found substitution tilings leading to *fault lines*: A, B are rectangles of height 1, with A of length $(1 + \sqrt{13})/2$ and B of length 3. The substitution is

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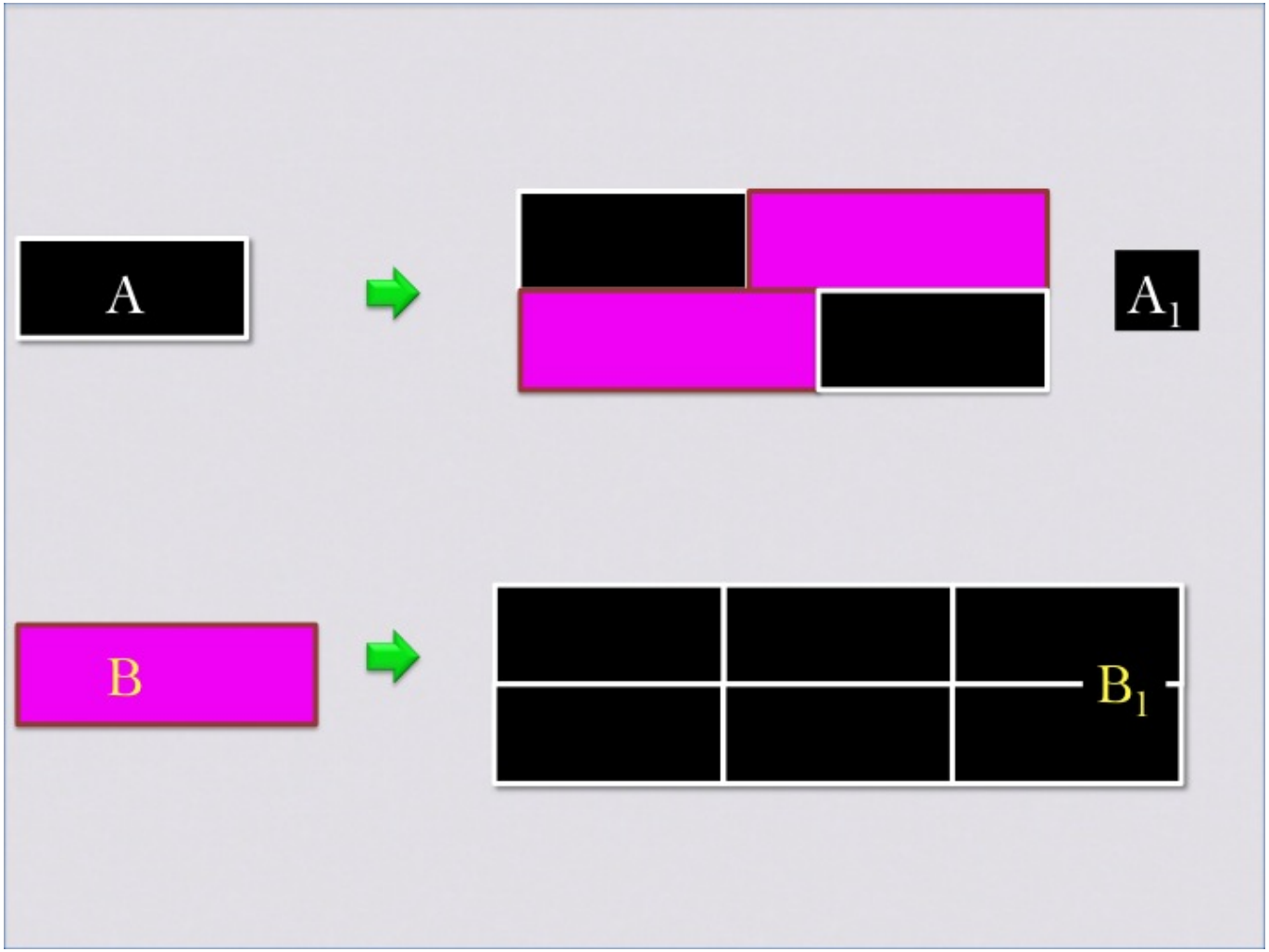

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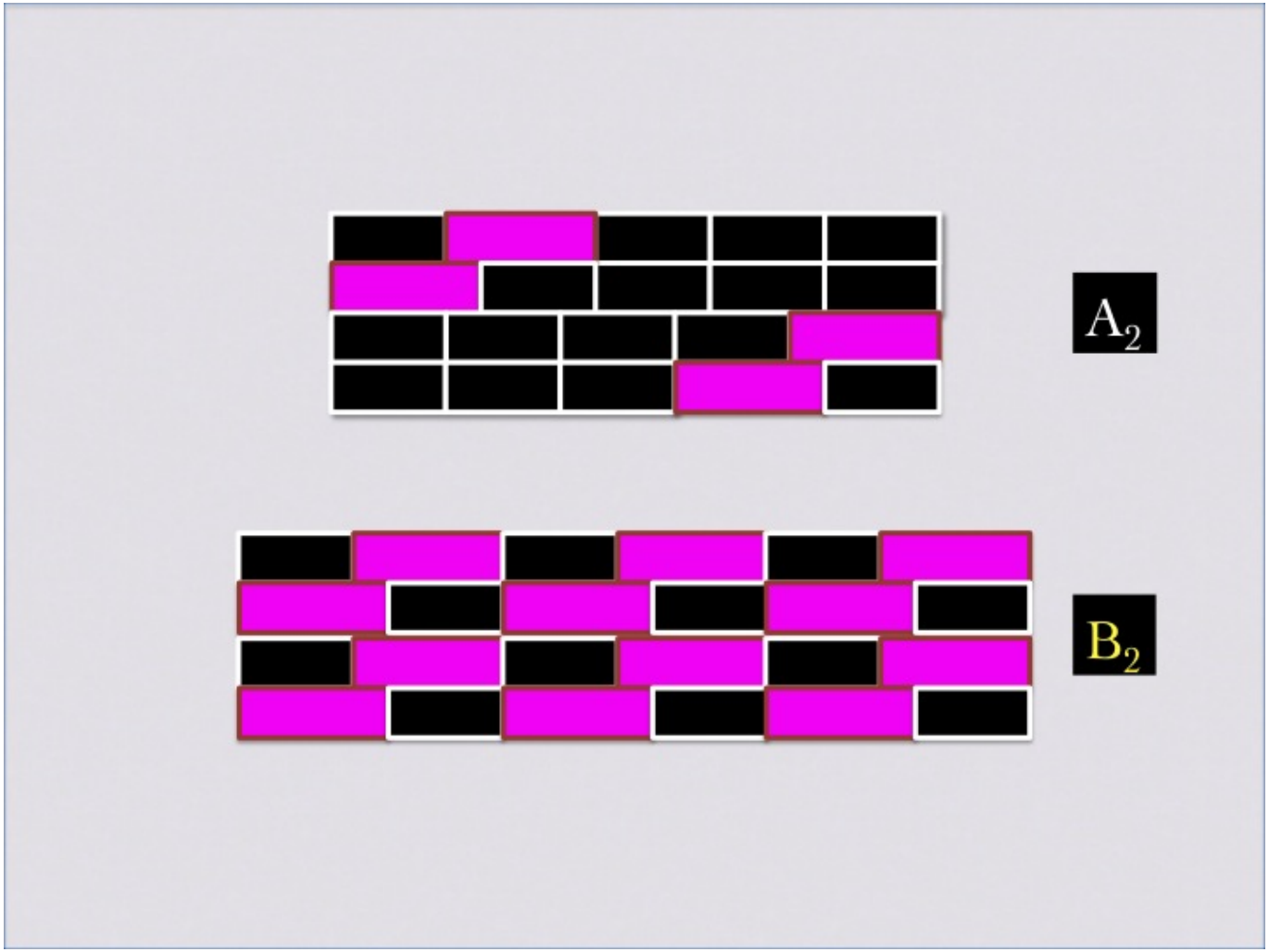

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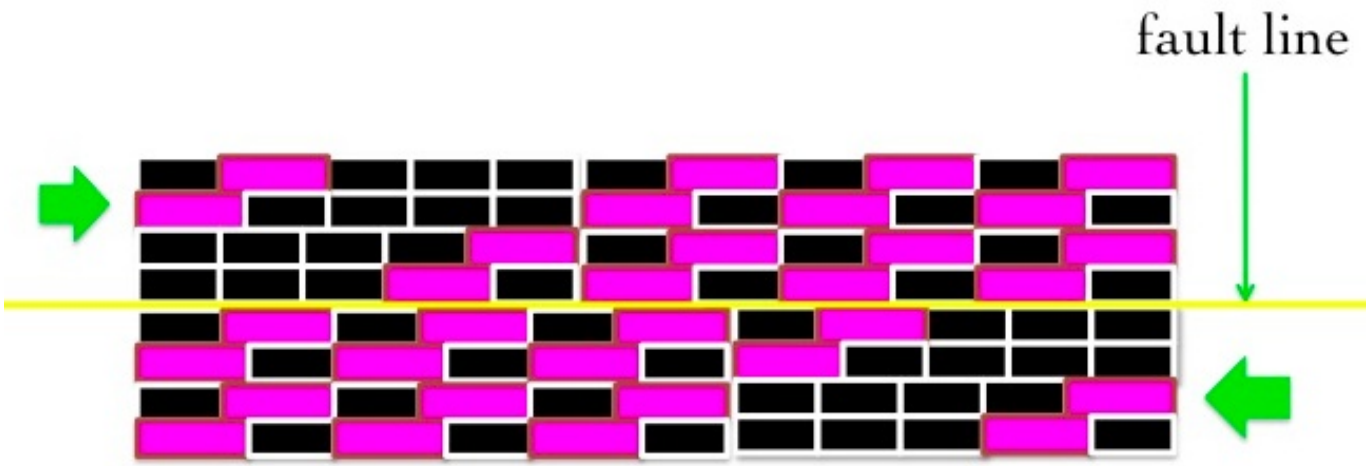
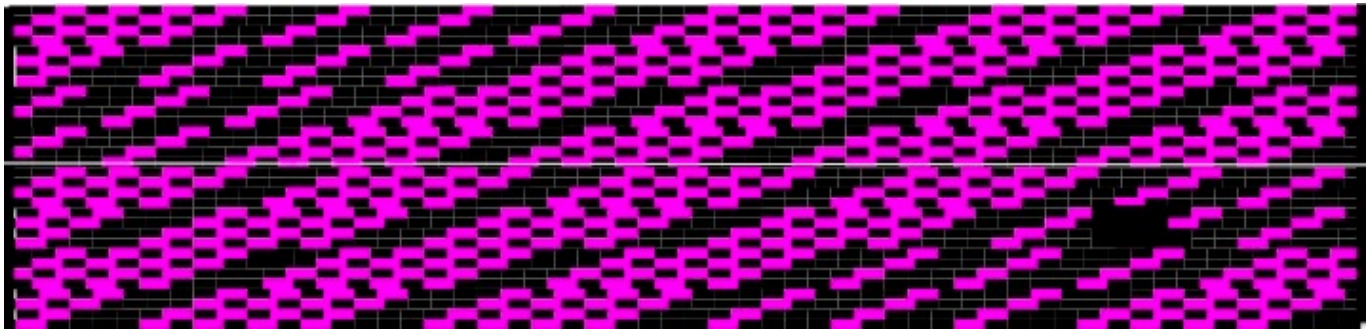
Fault lines are horizontal lines so that the parts of the tiling on either side can slide while giving another tiling of the tiling space.

Problem: *Can one describe the dynamics of this sliding by an extension of the Pearson Laplacian ?*

Fault Lines





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Problem: *Is there an extension of the Pearson Laplacian, or of diffuson on fractals, describing the dynamics of defects ?*

It’s time to take off !!