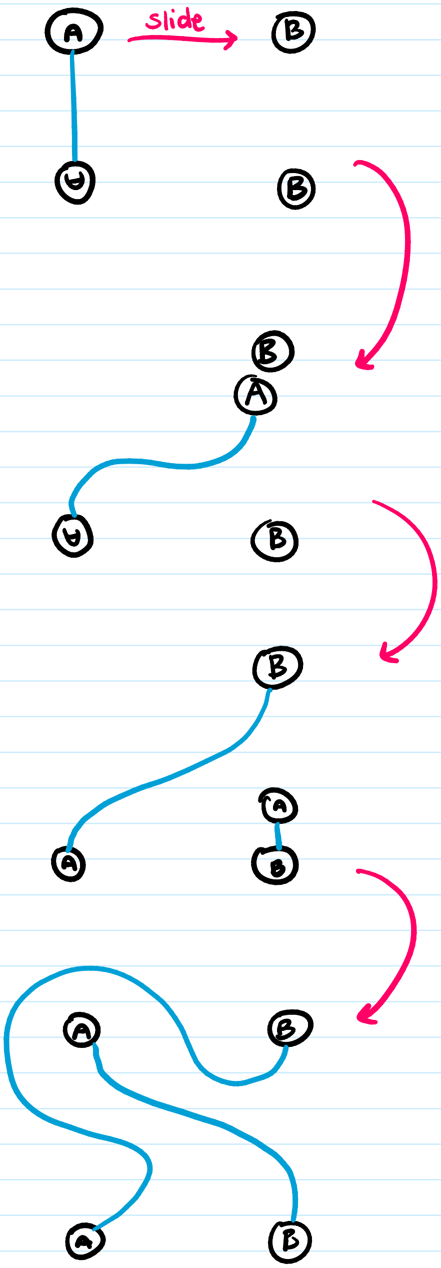
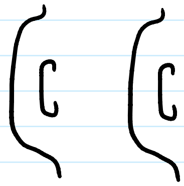
Week 9, 10/17 and 10/19

Tuesday, October 17, 2023 12:39 PM

Last time:



3-d 1-handle



H_2 and intersection form when X^4 has 1-handles:

2-handles attached along K_i

$\alpha = \sum a_i K_i$ is a linear combination representing a class $[\alpha]$ in $H_1(X)$

$a_i \in \mathbb{Z}$

If $[\alpha] = 0$, then α is null-homologous, then

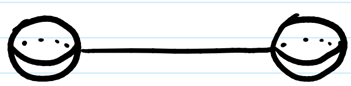
by taking cores of 2-handles together with a pushed-in Seifert surface, can think of $\alpha \in H_2(X)$

When X has 3-handles:

quotient by a subspace.

Exercise: use this idea to give a description of H_2 and the intersection form.

Handle Cancellation:

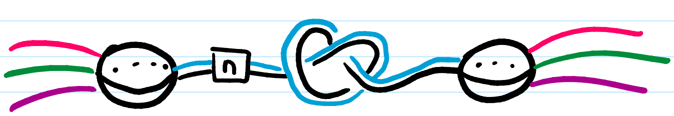


$\updownarrow$ isotopy

What if other 2-handles run over the 1-handle?

say:

with framing:

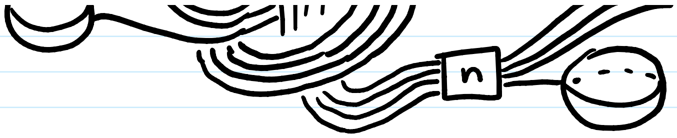


double strand notation for framing

idea:

handleslide over 2-handle

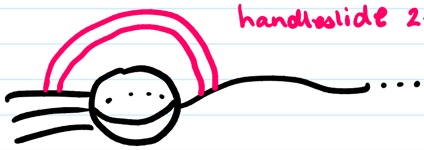
Now the 1-handle and a 2-handle cancel

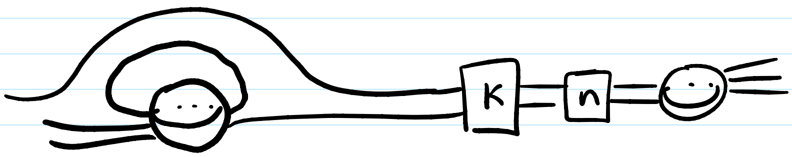


a 2-handle
cancel

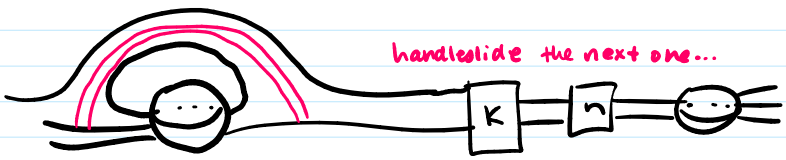
how:

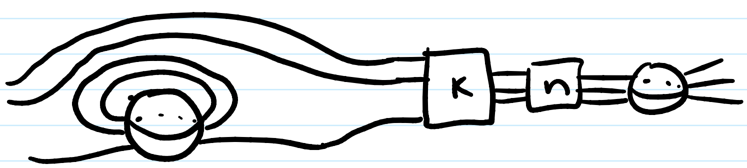
handleslide 2-handle over 2-h



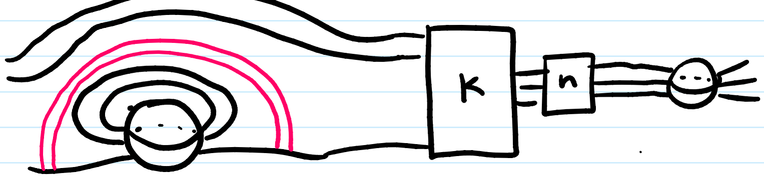


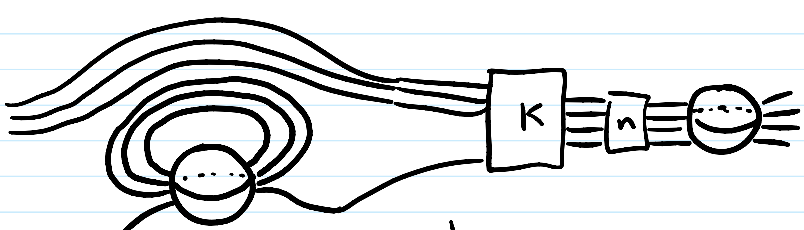
handleslide the next one...



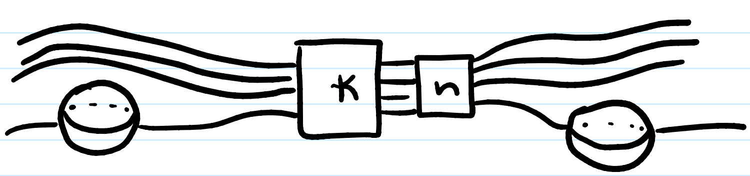


handleslide the last one...



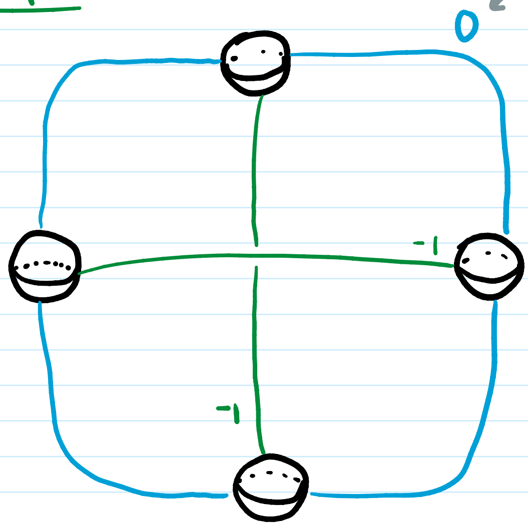


$\downarrow$ isotopy



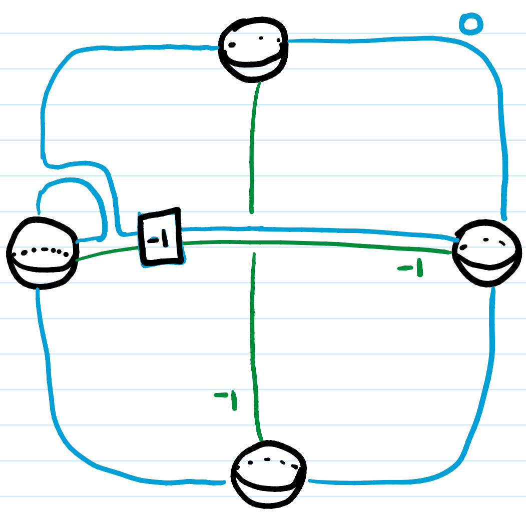
Example:

Example:



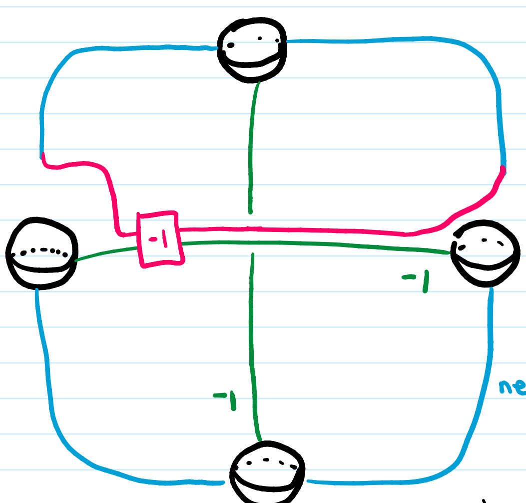
(relative to blackboard framing)

Want to do handleslides and see if we can cancel.
Slide 0-framed 2-h over (-1)-framed 2-handle:



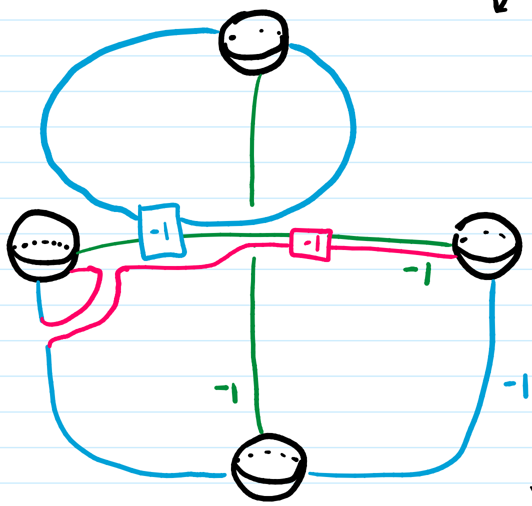
pull along the parallel strand

isotopy

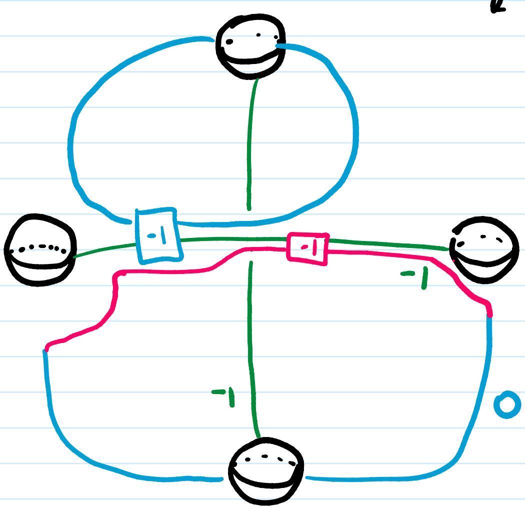


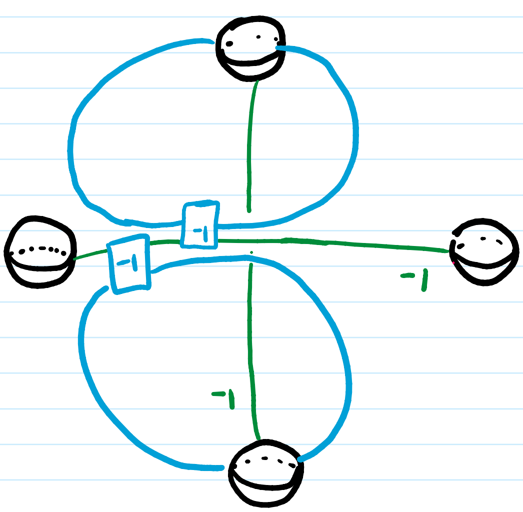
new framing: $(0)+(-1)+2(0)$
$=-1$

handleslide

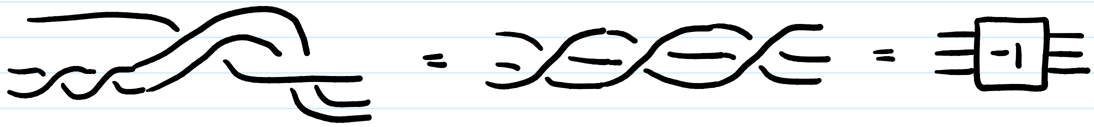


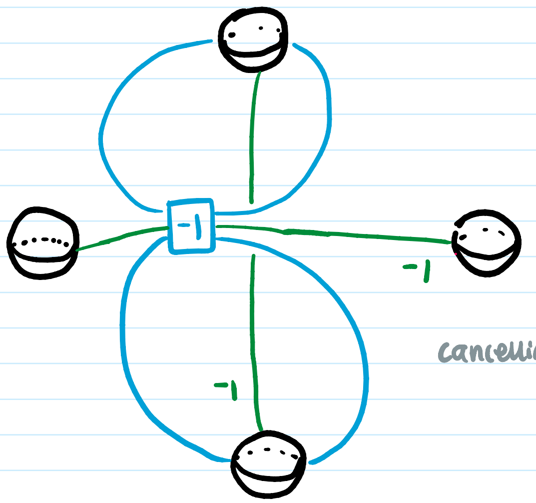
isotopy



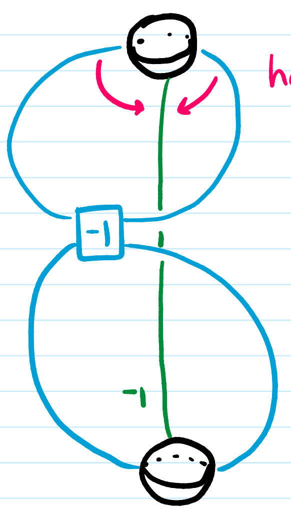


Remark:

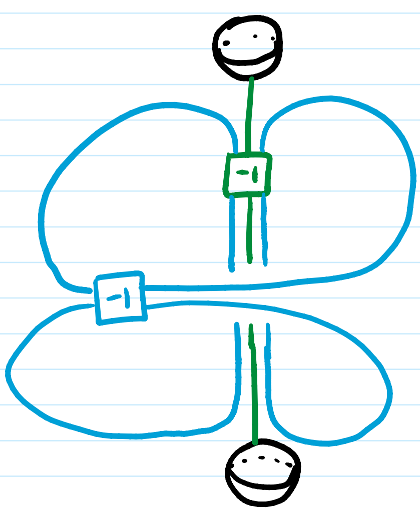




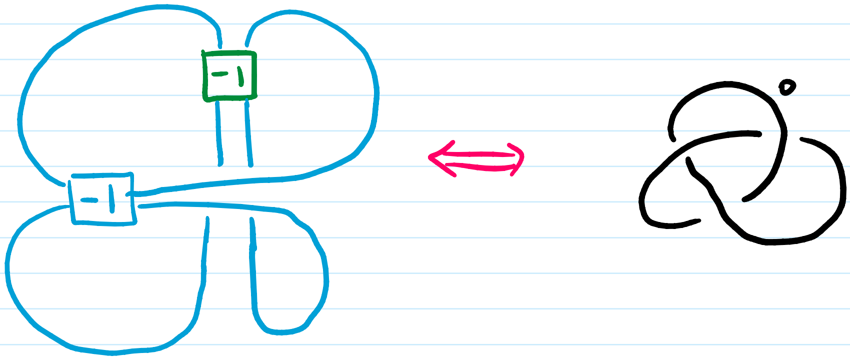
cancelling 1/2 pair



handleslide



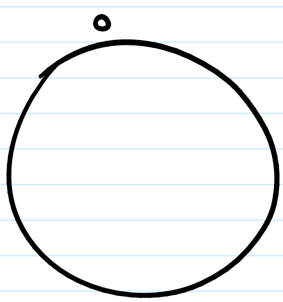
1/2 cancelling pair



Exercise: this will return to 

(done above)

Example: 2-3 cancelling pair



$\cup$ (3-h)

$$\partial(\underbrace{0\text{-h} \cup 2\text{-h}}_{S^2\times D^2}) = \partial(S^2\times D^2) = S^2\times S^1$$

3-h $D^3\times D^1$

attaching sphere $S^2\times\{pt\}$ intersects belt sphere ($\{pt\}\times S^1$) of 2-handle in a unique point.

reference: Ch 5 Gompf & Stipsicz

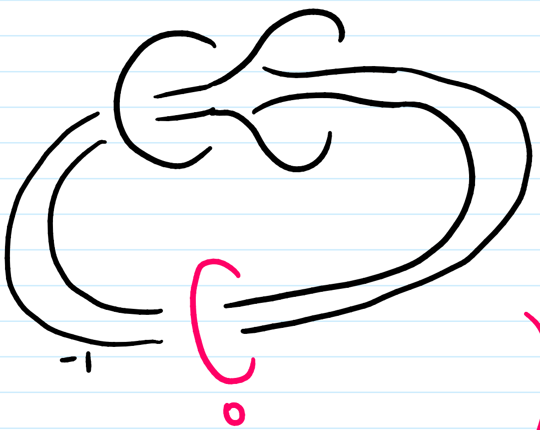
Proposition

Proposition

Let X be a closed 4-mfd described by a Kirby diagram. Then a 3-handle can be cancelled (after handleslides) iff it is possible to slide 2-handles to obtain a 0-framed unknot isolated from the rest of the diagram.

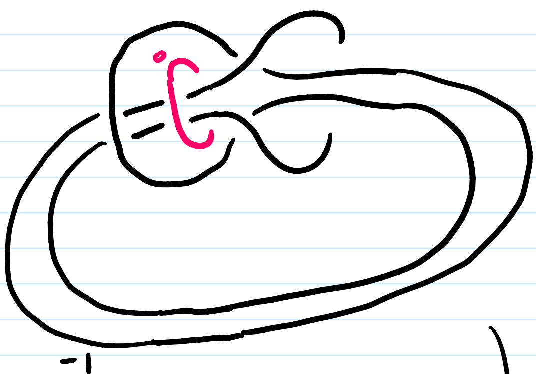
Remark: In practice, can be hard to use this, since it doesn't tell us how to slide 2-handles.

Example:



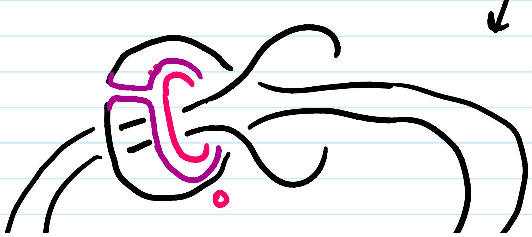
$\cup$ (3-h) $\cup$ (4-h)

isotopy



$\cup$ (3-h) $\cup$ (4-h)

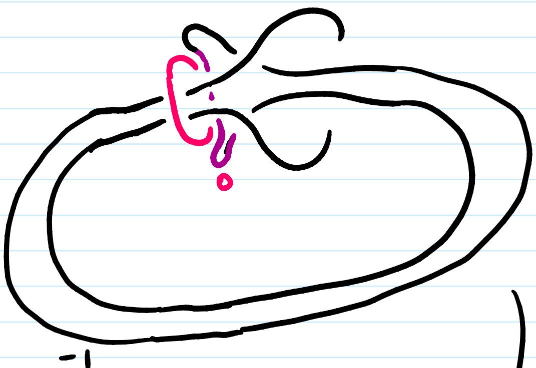
handleslide



$\cup$ (3-h) $\cup$ (4-h)

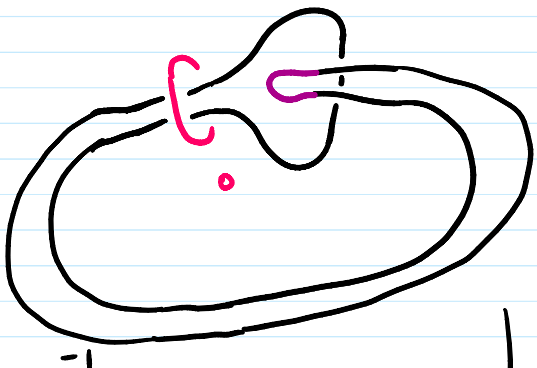
$\cup (3\text{-}h) \cup (4\text{-}h)$

$\downarrow$ isotopy



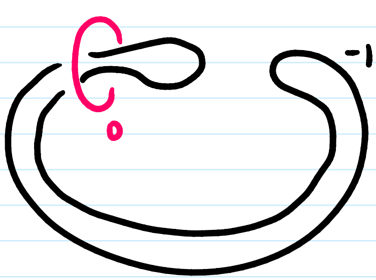
$\cup (3\text{-}h) \cup (4\text{-}h)$

$\downarrow$ isotopy



$\cup (3\text{-}h) \cup (4\text{-}h)$

$\downarrow$ isotopy



$\cup (3\text{-}h) \cup (4\text{-}h)$

$\downarrow$ isotopy

$\cup (3\text{-}h) \cup (4\text{-}h)$

$\Big\downarrow$ 2/3 cancellation

$\cup\ (4-n)$

this is $\overline{\mathbb{C}P}^2$

Exercise:

There are some nice exercises in Gompf & Stipsicz

-Exercises 5.1.10 b-d

relating 4-mfds to group theory and Andrew-Curtis conjecture

which is a group theory conjecture.

If you have a Group presentation for trivial group with same # of generators and relations (called balanced) it can be reduced to empty presentation by

1. inversion and permutation of generators and relations,
2. conjugation of relations by generators
3. multiplying a generator (resp. relation) by another
4. adding or deleting a gen. g together with relation g

these are called the Andrew-Curtis moves.

Exercise:

Let X be a simply connected closed 4-mfd described by a Kirby diagram w/ no 3-handles. Describe a presentation for $\pi_1(X)$ (Is it balanced?)

What effect do 1-handle slides on this presentation?

What effect do 1-handle slides on this presentation?

2-handleslides?

1/2 cancelling pair?

Compare to Andrew Curtis moves.

Remark on Cancelling handles:

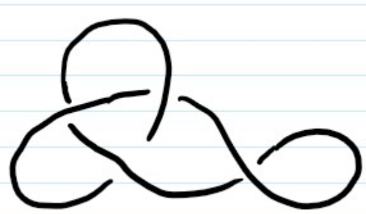
1) Formula for framing for 2-handleslides works when 2-h's are attached to D^4

2) Double strand notation works in general

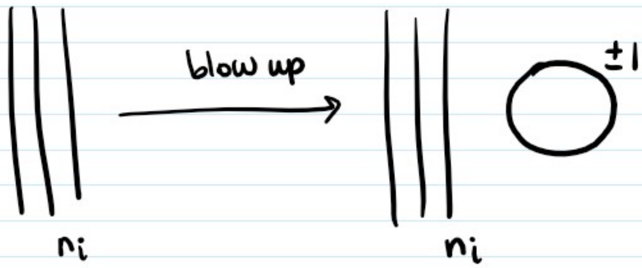
Example: blackboard framing

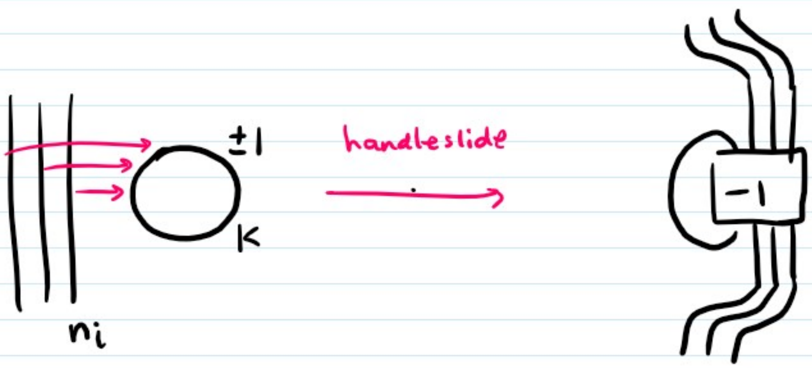
(depends on projection)

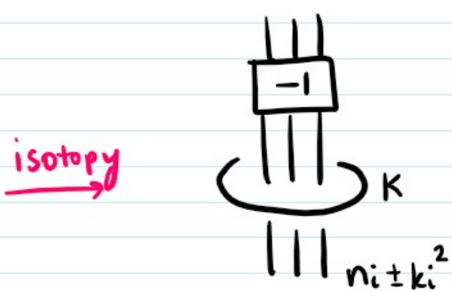
different than the blackboard framing of



BLOWING-UP $\# \overline{\mathbb{CP}^2}$ or $\# \mathbb{CP}^2$







k_i = signed count of # of strands of K_i in bunch

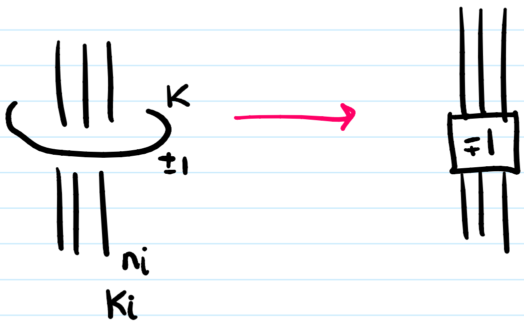
$$(\alpha_i \pm k_i \alpha)^2$$
$$= \alpha_i^2 + k_i^2 \alpha^2 \pm 2\,lk(K_i, K)$$
$$= n_i \pm k_i^2$$

α_i = class in H_2 associated to K_i

α = class of H_2 associated to K

(assuming they're null-homologous)

Can do the opposite



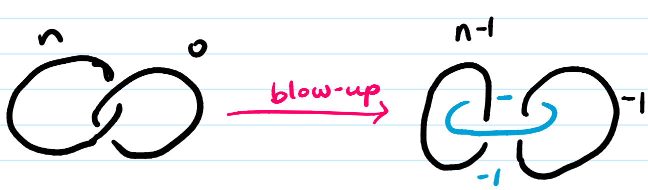
Remarks:

① any ±1 framed unknot in a Kirby diagram (possibly linking other components) represents $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$ summed

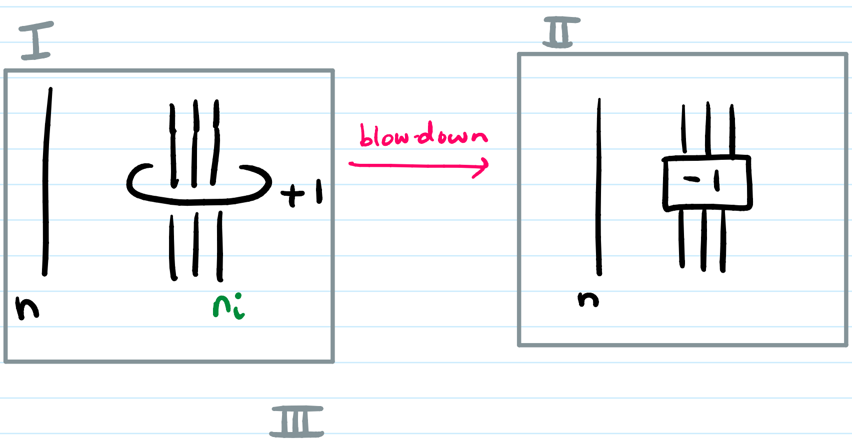
② Blowing up (or down) doesn't change the boundary of the 4 mfd

Examples:

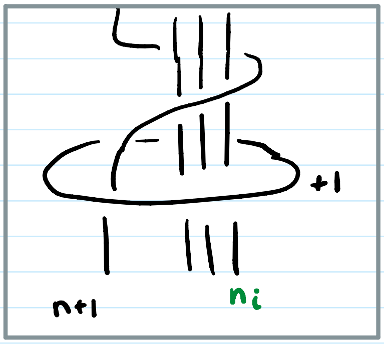
①



②



$\xrightarrow{\text{blow-up}}$



NOTE: $\quad \text{I} \xrightarrow{\text{blow-down}} \text{II} \xrightarrow{\text{blow-up}} \text{III}$

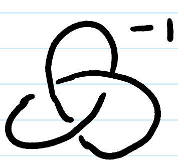
is the same as

$\text{I} \xrightarrow[\text{+1 framed unknot}]{\text{handleslide over}} \text{III}$

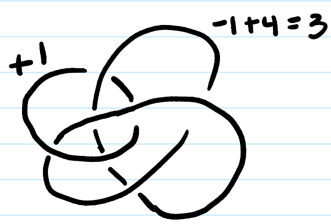
That is...

handleslide (in S^3) over a (+1)-framed unknot can be obtained by a blow-down followed by a blow-up

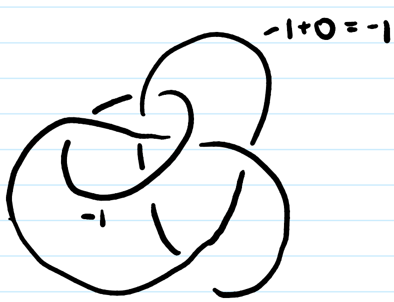
Example:



$\xrightarrow[\text{up}]{\text{blow}}$



blow
up

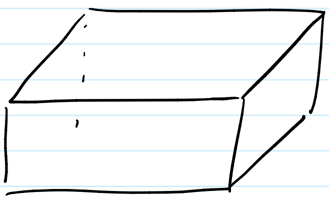


Exercise: sit, stare, convince yourself

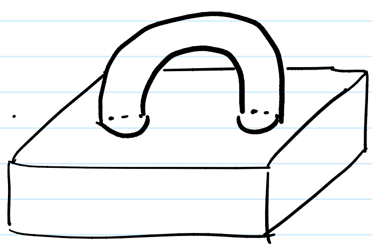
Exercise: (Gompf-Stipsicz, Ex 5.1.12b)

Let L and L' be framed links in S^3 and suppose L' is obtained from L by a handleslide. Prove that L' can also be obtained from L by a sequence of blowups and blow-downs.

1-Handles Revisited:



M



$M \cup (1\text{-}h)$

same as

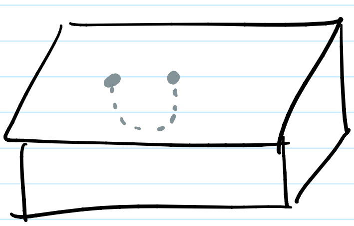
think of as 1/2 canceling pair

M

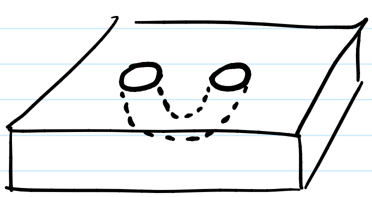
homeomorphic

$M - (2\text{-}h)$

homeomorphic



M



M w/ drilled nbhd of arc

Upshot:

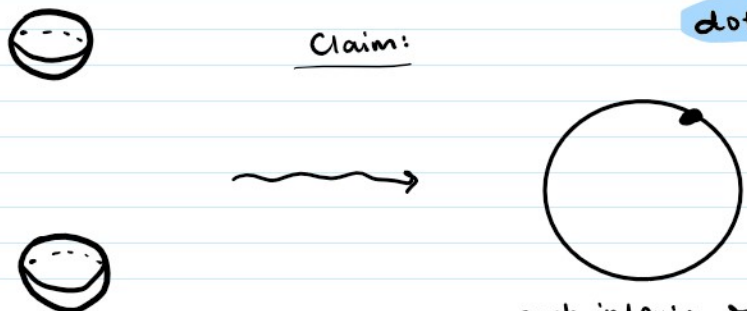
Adding a 1-handle is the same as pushing the interior of an interval D into int(M) and then removing a tubular nbhd of D

co-core of the 2-handle

Same idea for 4-dimensional 1-handles:

Claim:

dotted-circle notation

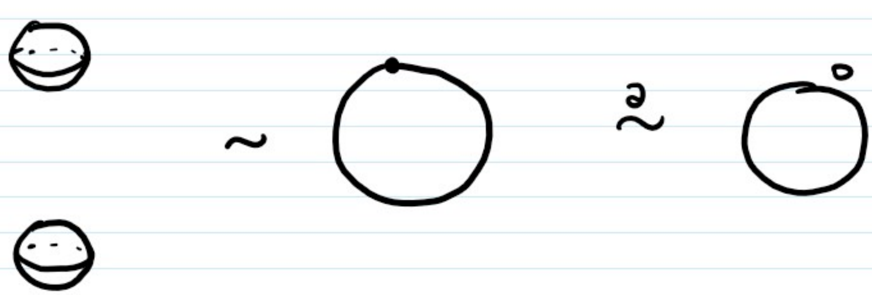


$S^1 \times D^3$

push interior of disk D into int(M) then remove tubular nbhd of D

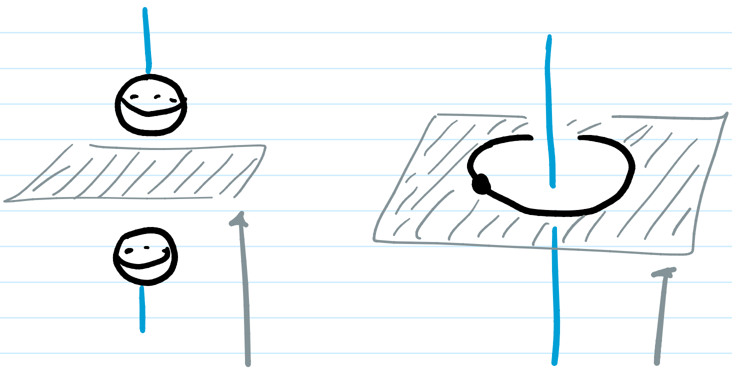
Exercise: a 4-ball $B^4 - (\text{pushed in disk})$ is $S^1 \times D^3$

So,



Example:

Let's compare some curves and surfaces in these two pictures:



plane between them is an S^2
and intersects at ∞