Surfaces and their symmetries

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References can be found at: http://bit.ly/surfsymmlinks
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2. Mapping class groups
3. Structures on surfaces
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       2. Geometry
       3. Topology
       4. Dynamics
Pt I. 1. Classification of surfaces  Riemann
2. Mapping class groups  Dehn
3. Structures on surfaces  Series

Pt II. 1. Group theory  Klein
2. Geometry  Thurston
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Riemann

Felix Klein

Max Dehn

J.W. Alexander

Joan Birman

Bill Thurston

Caroline Series
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Bernhard Riemann (1826-1866)

Riemann sum
Riemann hypothesis
Riemann surface
Riemannian manifold
topological open disks
topological open disks

\[ \cong \quad \cong \quad \cong \quad \cong \quad \cong \]

\[ \cong \quad \mathbb{R}^2 \]
topological open disks

\[ \cong \quad \cong \quad \cong \quad \cong \]

\[ \cong \quad \mathbb{R}^2 \quad \times \quad \exists \]}
Surfaces: “locally $\mathbb{R}^2$"
Surfaces: “locally $\mathbb{R}^2$"
Surfaces: “locally $\mathbb{R}^2$"
Surfaces: “locally $\mathbb{R}^2$"
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- Begin with a rectangle with side $A$ to be glued to side $a$, and $B$ to $b$.
- Roll the figure to bring $A$ to $a$.
- Glue $A$ to $a$, forming a cylinder.
- Bend cylinder around to bring $B$ in contact with $b$.
- Glue $B$ to $b$, forming a doughnut-shaped surface or torus.
The classification of surfaces
The classification of surfaces

The compact, connected, orientable two-dimensional manifolds are...
The classification of surfaces

The compact, connected, orientable two-dimensional manifolds are...
The classification of surfaces

Could add/drop adjectives: with boundary, with punctures, nonorientable, “big”
homeomorphic
Find the homeomorphism

Goofy Torus Day
March 7
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Dehn invariant

Dehn surgery

Dehn twist

Dehn–Lickorish theorem

Max Dehn
(1878-1952)
Dehn twist
\[ \text{Mod}(S_g) = \text{Homeo}^+(S_g)/\text{homotopy} \]
$S_g$

\[ \text{Mod}(S_g) = \text{Homeo}^+(S_g)/\text{homotopy} \]

\[ \text{Mod}(A) \text{ is } \mathbb{Z} \]
$S_g$

$\text{Mod}(S_g) = \text{Homeo}^+(S_g)/\text{homotopy}$

Dehn twist
Theorem (Dehn–Lickorish)
Finitely many Dehn twists generate $\text{Mod}(S_g)$. 
Theorem (Dehn–Lickorish)
Finitely many Dehn twists generate $\text{Mod}(S_g)$.

Theorem (Humphries, 1979)
For $g \geq 2$, $2g + 1$ Dehn twists generate $\text{Mod}(S_g)$, and this is sharp.
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Caroline Series (1951-present)  

Birman–Series theorem
“ambient” Riemannian geometry

constant curvature geometry
 (spherical, Euclidean, hyperbolic)

singular flat geometry
“ambient” Riemannian geometry

constant curvature geometry
  (spherical, Euclidean, hyperbolic)

singular flat geometry

conformal, algebraic, projective, foliations...
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Felix Klein (1849-1925)

Klein model of $\mathbb{H}^2$

Klein bottle

Klein quartic

Klein 4-group
Klein model of $\mathbb{H}^2$

Klein bottle

Klein quartic

Klein 4-group

Felix Klein
(1849-1925)
Klein quartic

\[ x^3y + y^3z + z^3x = 0 \]
Klein quartic

\[ x^3y + y^3z + z^3x = 0 \]
Klein quartic

\[ x^3y + y^3z + z^3x = 0 \]

isometry group of order 168
A Primer on Mapping Class Groups

BENSON FARB
DAN MARGALIT
7.5 GENERATING THE MAPPING CLASS GROUP WITH TORSION

We conclude this chapter with the following curious theorem of Feng Luo [132]. By an *involution* in a group we simply mean any element of order 2.

**Theorem 7.16** For $g \geq 3$, the group $\text{Mod}(S_g)$ is generated by finitely many involutions.
7.5 GENERATING THE MAPPING CLASS GROUP WITH TORSION

We conclude this chapter with the following curious theorem of Feng Luo [132]. By an *involution* in a group we simply mean any element of order 2.

**Theorem 7.16** For $g \geq 3$, the group $\text{Mod}(S_g)$ is generated by finitely many involutions.

Every mapping class group is generated by 6 involutions

Tara Brendle and Benson Farb
Problem: For $k > 2$, can $\text{Mod}(S_g)$ be generated by elements of order $k$? How few?
Obstacle:
When do higher-order elements even exist in $\text{Mod}(S_g)$?
Orders of torsion elements in $\text{Mod}(S_3)$:

1, 2, 3, 4, 6, 7, 8, 9, 12, 14
Number theoretic conditions for the existence of torsion elements in Mod(S_g)

(i) (the Hurwitz formula) \( 2(g - 1)/n = 2(g' - 1) + \sum_{i=1}^{l}(1 - 1/\lambda_i) \).

(ii) (Nielsen [Ni1, (4.6)]) \( \sum_{i=1}^{l} \sigma_i/\lambda_i \) is an integer.

(iii) (Wiman [W]) \( n \leq 4g + 2 \).

(iv) (Harvey [H]) Assume \( g \geq 2 \). Set \( M = \text{lcm}(\lambda_1, \ldots, \lambda_l) \). Then we have:

1. \( \text{lcm}(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_l) = M \) for all \( i \), where \( \lambda_i \) denotes the omission of \( \lambda_i \).
2. \( M \) divides \( n \), and if \( g' = 0 \), then \( M = n \).
3. \( l \neq 1 \), and, if \( g' = 0 \), then \( l \geq 3 \).
4. If \( 2|M \), the number of \( \lambda_1, \ldots, \lambda_l \) which are divisible by the maximal power of 2 dividing \( M \) is even.

(Ashikagana and Ishizaka)
Theorem (L., 2017)

Let $k \geq 6$ and $g \geq (k - 1)^2 + 1$. Then $\text{Mod}(S_g)$ is generated by three elements of order $k$.

Also, $\text{Mod}(S_g)$ is generated by four elements of order 5 when $g \geq 8$. 
$k = 5$

genus 5

genus 4
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Bill Thurston (1946-2012)

Thurston map
Thurston norm
Thurston geometry
Nielsen–Thurston classification
topological object

nice geometric structure?

yes!

no.

break into pieces

Thurston
closed oriented 3-manifold

nice geometric structure?

yes!
one of eight geometries

no.

reduce along spheres and tori

The Geometrization Theorem

Thurston Perelman
mapping class

nice geometric structure?

yes!

periodic or pseudo-Anosov

no.

reduce along an invariant multicurve

Nielsen–Thurston classification of mapping classes

Thurston

Nielsen
The Nielsen-Thurston Classification

periodic

reducible
The Nielsen-Thurston Classification

- periodic
- reducible
- pseudo-Anosov
The Nielsen-Thurston Classification

- periodic
- reducible
- pseudo-Anosov
The Nielsen-Thurston Classification

periodic

reducible

pseudo-Anosov
normal closure
\[ \langle \langle g \rangle \rangle := \langle \text{conjugates of } g \rangle \]

normal generator
\[ \langle \langle g \rangle \rangle = G \]

symmetric groups: \[ \langle \langle (12) \rangle \rangle = S_n \]
braid groups: \[ \langle \langle \sigma_1 \rangle \rangle = B_n \]
Problem:
Characterize the mapping classes that normally generate $\text{Mod}(S_g)$. 
reducible elements

\[ \langle \langle T_c \rangle \rangle = \text{Mod}(S_g) \]

\[ \langle \langle T_d \rangle \rangle \neq \text{Mod}(S_g) \]
The Nielsen-Thurston Classification

Question: Are there pseudo-Anosov normal generators? (Long, 1986)
Question: Are there pseudo-Anosov normal generators?

(Long, 1986)
Theorem (L.–Margalit)
Every pseudo-Anosov with stretch factor less than $\sqrt{2}$ is a normal generator for Mod(S_g).
$f$ with stretch factor less than $\sqrt{2}$ \quad \iff \quad \text{short curve } c, \\
i(c, f(c)) \leq 2, \\
i(c, f^2(c)) \leq 2

(Farb-Leininger-Margalit, 2006)
Proof sketch

\[ i(c, f(c)) \leq 2 \]
Proof sketch

\[ i(c, f(c)) \leq 2 \]

\( c \) nonseparating:
\[ i(c, f(c)) = 0, \text{ union nonseparating} \]
\[ i(c, f(c)) = 0, \text{ union separating} \]
\[ i(c, f(c)) = 1 \]
\[ i(c, f(c)) = 2 \]

\( c \) separating:
\[ i(c, f(c)) = 0 \]
\[ i(c, f(c)) = 2 \]
Proof sketch

\[ i(c, f(c)) \leq 2 \]

\text{c nonseparating:}

- \( i(c, f(c)) = 0 \), union nonseparating
- \( i(c, f(c)) = 0 \), union separating
- \( i(c, f(c)) = 1 \)
- \( i(c, f(c)) = 2 \)

\text{c separating:}

- \( i(c, f(c)) = 0 \)
- \( i(c, f(c)) = 0 \)
- \( i(c, f(c)) = 2 \)
Obstacle: \([c] = [f(c)] \mod 2\)
How to enumerate triples \((c, f(c), f^2(c))\)?
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Alexander polynomial

Alexander trick

Alexander horned sphere

J. W. Alexander (1888-1971)

Alexander polynomial

Alexander trick

Alexander system
Alexander trick

\[ \text{Mod}(D) \text{ is trivial.} \]
Alexander trick

Mod(D) is trivial.

(stable) Alexander system
Alexander trick

Mod(D) is trivial.

(stable) Alexander system
Alexander trick

Mod(D) is trivial.

(stable) Alexander system

Mod(S_g) has trivial center for $g \geq 3$
Alexander systems for “big” surfaces

Theorem (Kerékjártó, Richards)
Connected, orientable surfaces are in one-to-one correspondence with triples \((g, \text{Ends}, \text{Ends}_g)\).
Alexander systems for “big” surfaces

Theorem (Hernández–Morales–Valdez)
Every big surface has a stable Alexander system.
Theorem (L.-Loving)
Every normal subgroup of Mod(S) has trivial center when S is big.
Theorem (L.–Loving)
Every normal subgroup of Mod(S) has trivial center when S is big.

Theorem (L.–Loving)
Every Mod(S) with S big contains a $\mathbb{Z} \rtimes \mathbb{Z}$ subgroup
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Birman–Hilden theory

Joan Birman (1927–present)

Birman exact sequence
Birman–Wenzel algebra
Birman–Series theorem
Theorem (Birman–Series)

The union of all simple closed geodesics on a finite-area hyperbolic surface is a set of Hausdorff dimension one; in particular this set is nowhere dense and has Lebesgue measure zero.
How to hear the shape of a billiard table

with Aaron Calderon, Solly Coles, Diana Davis, & Andre Oliveira
unfolding construction
bounce sequence:
…ACECBAEDBD…
bounce sequence: 
...ACECBAEDBD...

bounce spectrum: 
\( B(P) = \{\text{bounce sequences for } P\} \)
What geometric information about P can be recovered from information in B(P)?

bounce sequence: 
...ACECBAEDBD...

bounce spectrum: 
B(P) = {bounce sequences for P}
Must any two drum heads that have the same set of eigenfrequencies also have the same shape?
Context: Can one hear the shape of a drum? (Kac)

Must any two drum heads that have the same set of eigenfrequencies also have the same shape?

No.  
Gordon, Webb, and Wolpert
Orthogonal stretches of right-angled tables have identical bounce spectra.
Orthogonal stretches of right-angled tables have identical bounce spectra.

Theorem (Duchin, Erlandsson, Leininger, & Sadanand)
This is the only exception.
Otherwise, B(P) is a complete invariant of P.
Theorem (Calderon, Coles, Davis, Lanier, & Oliveira)
Adjacency of sides and sizes of angles can be reconstructed from $B(P)$. 
Adjacency: topology on $B(P)$

closure of $B(P)$ in $\Sigma^\mathbb{Z}$

common prefixes
convex vs. nonconvex

(a) Common prefixes $A, B$ that are not adjacent.
(a) Common prefixes $A, B$ that are not adjacent.
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Thanks!