Tropical Hypersurfaces in Economics, and the Unimodularity Theorem

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Material from ‘Tropical geometry to analyse demand’ (2012-14) and ‘Understanding preferences: “Demand types” and the existence of equilibrium with indivisibilities’ (2015)

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Geometric Analysis of Demand: Model

- $n$ indivisible goods.
\( n \) indivisible goods. Finite set \( A \subset \mathbb{Z}^n \) of bundles available.
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- Valuation $u : A \to \mathbb{R}$; quasilinear utility $u(x) - p \cdot x$

Example of $u(x)$
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- Agent demands bundles in set $D_u(p) = \arg \max_{x \in A} \{u(x) - p \cdot x\}$

**Example of $u(x)$**

![Diagram](attachment:image.png)
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- Investigate what is demanded where: study where demand changes.

**Example of $u(x)$**

\[ u(x) = x_1^2 x_2 \]

**Diagram:**

- A tropical hypersurface (TH) is defined as $\{ \text{prices } p \in \mathbb{R}^n \text{ where } \# D_u(p) > 1 \}$. 

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Example of \( u(x) \)

Definition: “Tropical Hypersurface (TH)”

\[ \mathcal{T}_u = \{ \text{prices } p \in \mathbb{R}^n \text{ where } \#D_u(p) > 1 \} \]
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Example of $u(x)$

Example of tropical hypersurface (TH)

Definition: “Tropical Hypersurface (TH)”

$$\mathcal{T}_u = \{ \text{prices } p \in \mathbb{R}^n \text{ where } \#D_u(p) > 1 \}.$$
Cells and facets

Definition

*Tropical hypersurface* $\mathcal{T}_u$: prices $p \in \mathbb{R}^n$ where $\#D_u(p) > 1$.

Definition

A *cell* of $\mathcal{T}_u$ is a non-empty set $\{p \in \mathcal{T}_u : X \subset D_u(p)\}$, where $|X| > 1$.

A *facet* is an $(n - 1)$-dimensional cell of $\mathcal{T}_u$.

The tropical hypersurface is the union of its facets.

In two dimensions, made up of line segments.
Cells and facets

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In 3 dimensions, made up of pieces of planes (facets / 2-cells), meeting along lines (1-cells).
Cells and facets

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A facet is an $(n - 1)$-dimensional cell of $\mathcal{T}_u$.

The tropical hypersurface is the union of its facets.

Definition

A unique demand region is a connected component of the complement of $\mathcal{T}_u$.

Equivalently, it is the non-empty (absolute) interior of a set of the form $\{p \in \mathcal{T}_u : x \in D_u(p)\}$.
A tropical hypersurface in $\mathbb{R}^n$ is the support of a rational polyhedral complex of pure dimension $(n - 1)$. 
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- Made of polyhedral pieces (intersections of finite collections of half-spaces): cells.
- Every face of a cell is a cell.
- The intersection of two cells is either $\emptyset$ or is a face of both the cells.
- $\mathcal{T}_u$ is contained in the union of the facets ($n - 1$-cells).
- Cells have rational slope.
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Recall a unique demand region is a connected component of the complement of $\mathcal{T}_u$

**Lemma**

$C \subseteq \mathcal{T}_u$ is a cell iff it is the intersection of the closures of a collection of unique demand regions around $\mathcal{T}_u$.

**Lemma**

Let $C$ be a cell of $\mathcal{T}_u$, and let $C^\circ$ be the relative interior of $C$. Then $D_u(p^\circ)$ is constant for $p^\circ \in C^\circ$. Moreover, the cell is the locus where this bundle is demanded: $C = \{p \in \mathbb{R}^n : D_u(p^\circ) \subseteq D_u(p)\}$. 
How does demand change as you cross a facet?

A tropical hypersurface is composed of **facets**: linear pieces in dimension \((n - 1)\).

If \(p\) is in a facet then the agent is indifferent between two bundles:

\[
u(x) - p.x = u(y) - p.y\]
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\begin{align*}
u(x) - p \cdot x &= u(y) - p \cdot y \\
\iff p \cdot (y - x) &= u(y) - u(x)
\end{align*}
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The change in bundle is in the direction normal to the facet.
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Change in bundle is minus ‘weight \(w\)’ times minimal facet normal.

Endow all facets with weights: weighted rational polyhedral complex.
Every tropical hypersurface is balanced: around each \((n - 2)\)-cell, \(\sum_i w_i v_i = 0\).

Theorem (Mikhalkin 2004)

A weighted rational polyhedral complex of pure dimension \((n - 1)\) has support equal to the tropical hypersurface of a valuation iff it is balanced.

- We need not write down valuations of discrete bundles.
- We can simply draw tropical hypersurfaces.

Project Aim understand economics via geometry.
Concavity

**Definition**

- A set $A \subseteq \mathbb{Z}^n$ is *discrete-convex* if $\text{Conv}(A) \cap \mathbb{Z}^n = A$.
- Write $\text{Conv}(u) : \text{Conv}(A) \to \mathbb{R}$ for the minimal weakly-concave function everywhere weakly greater than $u$.
- $u : A \to \mathbb{R}$ is concave if $A$ is discrete-convex and $u(x) = \text{Conv}(u)(x)$ for all $x \in A$. 

![Diagram showing concavity](https://example.com/concavity_diagram.png)
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Lemma

$u : A \rightarrow \mathbb{R}$ is concave $\iff D_u(p)$ is discrete-convex for all $p$
$\iff \forall x \in \text{Conv}(A) \cap \mathbb{Z}^n$ there exists $p \in \mathbb{R}^n$ with $x \in D_u(p)$.

This $U$ is not concave.

\[x_1 \quad 8 \quad 0 \quad 0\]
\[x_1 = 2 \quad x_1 = 1 \quad x_1 = 0\]
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This $U$ is not concave. $D_U(4)$ is not discrete-convex.
Concavity

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**Theorem (Mikhalkin, 2004)**

An $(n-1)$-dimensional balanced weighted rational polyhedral complex in $\mathbb{R}^n$ corresponds to an ‘essentially unique’ concave valuation $u$. 
Classifying valuations

Economists classify valuations by how agents see trade-offs between goods.

For divisible goods, ask how changes in each price affect each demand. Let \( x^\ast(p) \) be optimal demands of each good at a given price. 

\[ \frac{\partial x^\ast_i}{\partial p_j} > 0 \]

means goods are 'substitutes' (tea, coffee).

\[ \frac{\partial x^\ast_i}{\partial p_j} < 0 \]

means goods are 'complements' (coffee, milk).

With indivisible goods, start by crossing one facet.
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With indivisible goods, start by crossing one facet.
Economic properties from facets

Suppose every facet normal \( v \) to \( T_u \)…

has at most one +ve, one -ve coordinate entry.

\[
\left( \begin{array}{c}
-1 \\
2
\end{array} \right)
\]

\[
\left( x_1 + 1, x_2 - 2 \right)
\]

Decrease price \( i \) to cross a facet.
Economic properties from facets

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Decrease price \( i \) to cross a facet.

- Demand changes from \( x \) to \( x + v \), where \( v \) is a facet normal.
- By the law of demand, \( v_i > 0 \).

\[ \Rightarrow v_j \leq 0 \text{ for all } j \neq i. \]
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Demand more \((1, 4)\) bundles

\[ (0,0) \]

\[ (1,4) \]

\[ (2,8) \]

\[ (3,12) \]
Economic properties from facets

Suppose every facet normal $\mathbf{v}$ to $\mathcal{T}_u$... is in set $\mathcal{D} \subset \mathbb{Z}^n$.

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These facts define structure of trade-offs.

E.g. (car bodies, car wheels)
Economic properties from facets

Suppose every facet normal $\mathbf{v}$ to $\mathcal{T}_u$... is in set $\mathcal{D} \subset \mathbb{Z}^n$.

Definition: “Demand Type”

$u$ is of demand type $\mathcal{D}$ if every facet of $\mathcal{T}_u$ has normal in $\mathcal{D}$.

The demand type is the set of all such valuations.

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These facts define structure of trade-offs.
Recall that $T_u$ lives in price space. The dual space is quantity space.

Sets $\text{Conv} D_u(p)$ form a rational polyhedral complex, dimension $\dim A$.

This is our ‘demand complex’ (=`subdivided Newton polytope`).
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\textbf{Lemma}

\(\hat{\sigma} \in \mathbb{R}^{n+1} \text{ is a face of the graph of } \text{Conv}(u) \text{ iff the projection of } \hat{\sigma} \text{ to its first } n \text{ coordinates is a cell of the demand complex.}\)
Duality: The Demand Complex

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Demand complex cells are dual to the cells of $T_u$.

- $k$-dimensional pieces $\leftrightarrow (n - k)$-dimensional pieces.
- $\sigma \subsetneq \sigma' \iff C_{\sigma'} \subsetneq C_{\sigma}$
- Linear spaces parallel to demand complex and corresp. tropical hypersurface cells are dual.
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**Lemma**

*Bundles which are not demand complex vertices are either never demanded or only demanded at prices corresp. to the demand complex cell they’re in.*
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*Bundles which are not demand complex vertices are either never demanded or only demanded at prices corresp. to the demand complex cell they’re in.*
Aggregate Demand

Many agents: \( j = 1, \ldots, m \), valuations \( u^j : A_j \to \mathbb{R} \).

**Definition (Standard)**

**Aggregate demand** at \( p \) is the Minkowski sum of individual demands:

\[
D_{u^1}(p) + \cdots + D_{u^m}(p)
\]

Write \( A := A^1 + \cdots + A^j \) and \( U : A \to \mathbb{R} \) so aggregate demand is \( D_U(p) \).

Not hard to see we can use:

\[
U(x) = \max \left\{ \sum_j u^j(x^j) \mid x^j \in A_j, \sum_j x^j = x \right\}.
\]
Aggregate Demand

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**Definition (Standard)**

If supply is \( x \), a **competitive equilibrium** among agents \( i \) consists of

- allocations \( x^i \) such that \( \sum_i x^i = x \).
- price \( p \) such that \( x^i \in D_{u^i}(p) \) for all \( i \).

\( x \in D_U(p) \) for some \( p \).
Aggregate Demand

Many agents: \( j = 1, \ldots, m \), valuations \( u^j : A_j \to \mathbb{R} \).

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\[ \left\{ \begin{array}{l}
x \in D_U(p) \text{ for some } p \\
\end{array} \right. \]

Refer to supply \( x \in \text{Conv}(A) = \text{Conv}(A^1 + \cdots + A^m) \) as **relevant** supply.

\[ \forall \text{ Eqm for some relevant supply} \iff D_U(p) \text{ not discrete-convex for some } p \]
\[ \iff U \text{ is not concave.} \]
Tropical hypersurface of aggregate demand

\[ DU(p) = Du^1(p) + \cdots + Du^m(p) \]

Easy to draw \( TU \),
Tropical hypersurface of aggregate demand

\[ D_U(p) = D_{u^1}(p) + \cdots + D_{u^m}(p) \]

Easy to draw \( T_U \), just superimpose individual tropical hypersurfaces.

Corollary

*If* \( u^1, \ldots, u^m \) *are of demand type* \( D \) *then so is* \( U \).*
Tropical hypersurface of aggregate demand

\[ D_U(p) = D_{u_1}(p) + \cdots + D_{u_m}(p) \]

Easy to draw \( T_U \), just superimpose individual tropical hypersurfaces.

Then what is \( D_U(p) \)?

- If \( p \notin T_U \), easy: use “facet normal \( \times \) weight = change in demand”.
- If \( p \in T_{u_i} \), only one \( i \), and individual valuations concave, also easy.
- Interesting case: \( p \in T_{u_i}, T_{u_j} \) for \( i \neq j \).
Tropical hypersurface of aggregate demand

\[ D_U(p) = D_{u^1}(p) + \cdots + D_{u^m}(p) \]

Easy to draw \( \mathcal{T}_U \), just superimpose individual tropical hypersurfaces.

Then what is \( D_U(p) \)?

- If \( p \notin \mathcal{T}_U \), easy: use “facet normal \( \times \) weight = change in demand”.
- If \( p \in \mathcal{T}_{u^i} \), only one \( i \), and individual valuations concave, also easy.
- Interesting case: \( p \in \mathcal{T}_{u^i}, \mathcal{T}_{u^j} \) for \( i \neq j \).

**Lemma**

*Equilibrium fails for concave \( u^1, u^2 \) and some supply \( x \in A^1 + A^2 \) iff \( D_u(p) \) is not discrete-convex for some \( p \in \mathcal{T}_{u^1} \cap \mathcal{T}_{u^2} \).*
Theorem (Kelso and Crawford 1982)

Suppose

- \( A^i = \{0, 1\}^n \) for all agents \( i \).
- \( u^i : A^i \to \mathbb{R} \) is a concave substitute valuation for all agents.
- Supply \( x \in \{0, 1\}^n \).

Then competitive equilibrium exists.
**Theorem (Milgrom and Strulovici 2009)**

Suppose

- **domain** \( A^i = A \), a fixed product of intervals, for all agents \( i \).
- \( u^i : A^i \rightarrow \mathbb{R} \) is a concave **strong substitute** valuation for all agents.
- **Supply** \( x \in A \).

Then competitive equilibrium exists.
Suppose

- domain \( A^i \subset \{-1, 0, 1\}^n \) for all agents \( i \).
- \( w^i : A^i \rightarrow \mathbb{R} \) is a concave ‘full’ substitute valuation for all agents.
- Supply \( x = 0 \).

Then competitive equilibrium exists.
Theorem (Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp 2013)

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Seek generalised result of this form:

Suppose we fix a ‘description’.

- Agents all have concave valuations of this description.
- ‘Relevant’ supply (in the convex hull of domain of aggregate demand).

Ask: does competitive equilibrium always exist?
Theorem (Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp 2013)

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Seek generalised result of this form:

Suppose we fix a demand type \( D \).

- Agents all have concave valuations of demand type \( D \).
- ‘Relevant’ supply (in the convex hull of domain of aggregate demand).

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Theorem (Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp 2013)

Suppose

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Then competitive equilibrium exists.

Seek generalised result of this form:

Suppose we fix a demand type \( \mathcal{D} \).

- Agents all have concave valuations of demand type \( \mathcal{D} \).
- ‘Relevant’ supply (in the convex hull of domain of aggregate demand).

Ask: does competitive equilibrium always exist?

Yes, iff \( \mathcal{D} \) has a certain property...
Demand types and equilibrium

Recall:

- Could only demand \((1, 1)\) at price corresp. to the square demand complex cell.
- But at this price,
  - Red demands \((1, 0)\) or \((0, 1)\)
  - Blue demands \((0, 0)\) or \((1, 1)\)
- Switching choices takes us between the vertices of the square.
- There is no way to get to the bundle in the middle.
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Demand types and equilibrium

- The problem is that the bundle is in the middle of the square.
- There exists a bundle there because the area of the square is $> 1$.
- The area is (abs. value of) the determinant of vectors along its edges.

$$\det \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 2$$

- Avoid problems iff all sets of $n$ demand type vectors have $\det \pm 1$ or 0.

$\Rightarrow$ “unimodularity”.

*When vectors in $D$ span $\mathbb{R}^n$, unimodularity $\iff$ all sets of $n$ vectors have $\det \pm 1$ or 0.
Theorem (cf. Danilov, Koshevoy and Murota, 2001)

Fix a set $\mathcal{D} \subseteq \mathbb{Z}^n$. A competitive equilibrium exists for
- every finite set of agents with concave valuations of type $\mathcal{D}$
- any relevant supply

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*When vectors in $\mathcal{D}$ span $\mathbb{R}^n$, unimodularity $\Leftrightarrow$ all sets of $n$ vectors have $\det \pm 1$ or $0$. 
Demand types and equilibrium

**Theorem (cf. Danilov, Koshevoy and Murota, 2001)**

Fix a set $\mathcal{D} \subseteq \mathbb{Z}^n$. A competitive equilibrium exists for

- every finite set of agents with concave valuations of type $\mathcal{D}$
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**iff** $\mathcal{D}$ is unimodular.

From this, follows existence of equilibrium in:

- Strong substitutes / $M^\mathbb{R}$-concave valuations (Danilov et al., 2003, Discrete Applied Math., Milgrom and Strulovici, 2009, JET).
- Gross substitutes and complements (Sun and Yang, 2006, Ecta).
- Full substitutability on a trading network (Hatfield et al. 2013, JPE).

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Demand types and equilibrium

**Theorem (cf. Danilov, Koshevoy and Murota, 2001)**

Fix a set \( \mathcal{D} \subseteq \mathbb{Z}^n \). A competitive equilibrium exists for

- every pair of agents with concave valuations of type \( \mathcal{D} \)
- any relevant supply

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From this, follows existence of equilibrium in:

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Transverse Intersections

Definition

$\mathcal{T}_u^1, \mathcal{T}_u^2$ intersect transversally at $p$ if for minimal $C_1, C_2$ of $\mathcal{T}_u^1, \mathcal{T}_u^2$ containing $p$, have $\dim(C_1 + C_2) = n$. 

Lemma (See e.g. Maclagan and Sturmfels 2015)

$\mathcal{T}_u^1, \mathcal{T}_u^2$ intersect transversally after a small generic translation.

Translations in valuations: If $u \in (x) = u(x) + \epsilon v$, then $\mathcal{T}_u^1 = \mathcal{T}_u^1 + \{\epsilon v\}$.
**Definition**

\( \mathcal{T}_{u_1}, \mathcal{T}_{u_2} \) intersect *transversally* at \( p \) if for minimal \( C_1, C_2 \) of \( \mathcal{T}_{u_1}, \mathcal{T}_{u_2} \) containing \( p \), have \( \dim(C_1 + C_2) = n \).
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$\mathcal{T}_{u^1}, \mathcal{T}_{u^2}$ intersect transversally after a small generic translation.

Translations in valuations: If $u_\epsilon(x) = u(x) + \epsilon v \cdot x$ then $\mathcal{T}_{u_\epsilon} = \mathcal{T}_{u} + \{\epsilon v\}$. 
Proposition

If equilibrium does not exist for valuations $u^1$ and $u^2$, and for some relevant supply $y$, then for any $v \in \mathbb{R}^n$, equilibrium also fails for valuations $u^1$ and $u^2_\epsilon$, in which $u^2_\epsilon(x) = u^2(x) + \epsilon v \cdot x$, for supply $y$ and all sufficiently small $\epsilon > 0$. 
Proposition

If equilibrium does not exist for valuations \( u^1 \) and \( u^2 \), and for some relevant supply \( y \), then for any \( v \in \mathbb{R}^n \), equilibrium also fails for valuations \( u^1 \) and \( u^2 + \epsilon v \cdot x \), in which \( u^2 + \epsilon v \cdot x \) for supply \( y \) and all sufficiently small \( \epsilon > 0 \).

Proof. Let \( P^j(x) \) be prices \( p \) where \( x \in D_{u^j}(p) \).

Failure of equilibrium \( \Leftrightarrow \)
\[
P^1(x^1) \cap P^2(x^2) = \emptyset \quad \forall (x^1, x^2) \in A^1 \times A^2 \text{ s.t. } x^1 + x^2 = y.
\]
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If equilibrium does not exist for valuations $u^1$ and $u^2$, and for some relevant supply $y$, then for any $v \in \mathbb{R}^n$, equilibrium also fails for valuations $u^1$ and $u^2$, in which $u^2_\epsilon(x) = u^2(x) + \epsilon v \cdot x$, for supply $y$ and all sufficiently small $\epsilon > 0$.

Proof. Let $P^j(x)$ be prices $p$ where $x \in D_{wj}(p)$. Failure of equilibrium $\iff$

$$P^1(x^1) \cap P^2(x^2) = \emptyset \ \forall (x^1, x^2) \in A^1 \times A^2 \text{ s.t. } x^1 + x^2 = y.$$
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Clearly the prices $p$ where $x^2 \in D_{u^2_\epsilon}(p)$ are $\{\epsilon v\} + P^2(x^2)$. 
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If equilibrium does not exist for valuations $u^1$ and $u^2$, and for some relevant supply $y$, then for any $v \in \mathbb{R}^n$, equilibrium also fails for valuations $u^1$ and $u^2_\epsilon$, in which $u^2_\epsilon(x) = u^2(x) + \epsilon v \cdot x$, for supply $y$ and all sufficiently small $\epsilon > 0$.

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Clearly the prices $p$ where $x^2 \in D_{u^2_\epsilon}(p)$ are $\{\epsilon v\} + P^2(x^2)$.

$P^j(x)$ is a cell, the closure of a unique demand region, or $\emptyset$: a polyhedron. If two polyhedra are disjoint, after a sufficiently small translation in any direction, they are still disjoint.
Proposition

If equilibrium does not exist for valuations $u^1$ and $u^2$, and for some relevant supply $y$, then for any $v \in \mathbb{R}^n$, equilibrium also fails for valuations $u^1$ and $u^2_\epsilon$, in which $u^2_\epsilon(x) = u^2(x) + \epsilon v \cdot x$, for supply $y$ and all sufficiently small $\epsilon > 0$.

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$P^j(x)$ is a cell, the closure of a unique demand region, or $\emptyset$: a polyhedron.

If two polyhedra are disjoint, after a sufficiently small translation in any direction, they are still disjoint.

Pick $\epsilon$ so this holds for translation $\epsilon v$ and all $(x^1, x^2) \in A^1 \times A^2$ s.t. $x^1 + x^2 = y$. 
Proposition

If equilibrium does not exist for valuations \( u^1 \) and \( u^2 \), and for some relevant supply \( y \), then for any \( v \in \mathbb{R}^n \), equilibrium also fails for valuations \( u^1 \) and \( u^2_{\epsilon} \), in which \( u^2_{\epsilon}(x) = u^2(x) + \epsilon v \cdot x \), for supply \( y \) and all sufficiently small \( \epsilon > 0 \).

Corollary

Equilibrium exists for all finite sets of concave valuations of a demand type and all relevant supplies iff it exists for all finite sets of concave valuations of that type which intersect transversally.
Unimodularity

For a linearly independent set \( \{v^1, \ldots, v^s\} \) in \( \mathbb{Z}^n \), the following are equivalent:

1. \( \{v^1, \ldots, v^2\} \) are unimodular;
2. There exist \( v^{s+1}, \ldots, v^n \in \mathbb{Z}^n \) such that the determinant of \( v^1, \ldots, v^n \) is \( \pm 1 \).
3. The parallelepiped whose edges are these vectors, that is, \( \left\{ \sum_{j=1}^n \lambda^j v^j : \lambda^j \in [0, 1] \right\} \), contains no non-vertex integer point;
4. If \( x \in \mathbb{Z}^n \) and \( x = \sum_{j=1}^s \alpha^j v^j \) with \( \alpha^j \in \mathbb{R} \), then \( \alpha^j \in \mathbb{Z} \) for all \( j \).
Finite collections of agents:

- Let $v^1, \ldots, v^k \in D$. Fix $p \in \mathbb{R}^n$.
- For $j = 1, \ldots, k$ choose a concave valuation with facet normal $v^j$, with weight 1, and with $p$ in its interior.
- So $D_{uj}(p) = \{x^j, x^j + v^j\}$ for $j = 1, \ldots, k$ and some $x^j$.
- Thus $D_U(p) = \{x + \sum_j \delta^j v^j : \delta^j \in \{0, 1\}\}$, where $x = \sum_j x^j$.
- This is the vertices of a parallelepiped
  \[ \Rightarrow D_U(p) \text{ is discrete-convex iff } v^1, \ldots, v^k \text{ are unimodular.} \]

\[ \{(1, 1), (-1, 1)\} \text{ not a unimodular set.} \]

\[ D_U(p) \text{ not discrete-convex.} \]
Necessity in the Unimodularity Theorem

Finite collections of agents:

- Let $v^1, \ldots, v^k \in D$. Fix $p \in \mathbb{R}^n$.
- For $j = 1, \ldots, k$ choose a concave valuation with facet normal $v^j$, with weight 1, and with $p$ in its interior.
- So $D_{u^j}(p) = \{x^j, x^j + v^j\}$ for $j = 1, \ldots, k$ and some $x^j$.
- Thus $D_U(p) = \{x + \sum_j \delta^j v^j : \delta^j \in \{0, 1\}\}$, where $x = \sum_j x^j$.
- This is the vertices of a parallelepiped
  \[ \Rightarrow D_U(p) \text{ is discrete-convex iff } v^1, \ldots, v^k \text{ are unimodular.} \]

Only two agents

- Let $k$ be minimal such that $v^1, \ldots, v^k$ not unimodular.
- Choose valuations $u^1, \ldots, u^k$ as above and let $u^0$ be aggregate valuation for $u^1, \ldots, u^{k-1}$.
- $D_{u^0}(p)$ is discrete-convex, by minimality, so is concave on $D_{u^0}(p)$.
- but $D_{u^0}(p) + D_{u^k}(p) = D_U(p)$ which is non discrete-convex.
Transverse intersections and changes in demand

Consider the linear span of changes in demand.

**Definition**

$L_\sigma := \langle \{y - x : x, y \in \sigma\} \rangle_\mathbb{R}$ where $\sigma$ is a demand complex cell.

**Lemma (e.g. Gruber, 2007)**

$\exists$ a basis for $L_\sigma$ consisting of edges of $\sigma$ (i.e. facet normals in T.H.).
Consider the linear span of changes in demand.

**Definition**

\[ L_\sigma := \langle \{ y - x : x, y \in \sigma \} \rangle_{\mathbb{R}} \text{ where } \sigma \text{ is a demand complex cell.} \]

**Lemma (e.g. Gruber, 2007)**

\[ \exists \text{ a basis for } L_\sigma \text{ consisting of edges of } \sigma \text{ (i.e. facet normals in T.H.).} \]

For \( p \in \mathcal{T}_{u_1} \cap \mathcal{T}_{u_2} \), write \( \sigma^j, \sigma \) for \( \text{Conv} D_{u_j}(p), \text{Conv} D_U(p) \).

**Lemma**

\[ L_\sigma = L_{\sigma^1} \oplus L_{\sigma^2} \text{ iff intersection is transverse at } p \]
Transverse intersections and changes in demand

Consider the linear span of changes in demand.

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\[ L_\sigma := \langle \{ y - x : x, y \in \sigma \} \rangle_{\mathbb{R}} \]

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\[ \exists \text{ a basis for } L_\sigma \text{ consisting of edges of } \sigma \text{ (i.e. facet normals in T.H.).} \]

For \( p \in \mathcal{T}_{u1} \cap \mathcal{T}_{u2} \), write \( \sigma^j, \sigma \) for \( \text{Conv} D_{u_j}(p), \text{Conv} D_U(p) \).

Lemma

\[ L_\sigma = L_{\sigma 1} \oplus L_{\sigma 2} \text{ iff intersection is transverse at } p \]

Corollary

If intersection is transverse and demand type is unimodular, and if \( y, x \in \sigma \cap \mathbb{Z}^n \), can write \( y - x = z^1 + z^2 \) where \( z^j \in L_{\sigma j} \cap \mathbb{Z}^n, j = 1, 2. \)
Proof of Sufficiency in Unimodularity Theorem

Definition

\[ L_\sigma := \langle \{ y - x : x, y \in \sigma \} \rangle \mathbb{R} \] where \( \sigma \) is a demand complex cell.
Proof of Sufficiency in Unimodularity Theorem

Definition

\[ L_\sigma := \langle \{ y - x : x, y \in \sigma \} \rangle_\mathbb{R} \text{ where } \sigma \text{ is a demand complex cell.} \]

Suppose \( u^1, u^2 \) concave and \( p \in T_{u^1} \cap T_{u^2} \). Write \( \sigma^j, \sigma \) for \( \text{Conv}D_{u^j}(p), \text{Conv}D_U(p) \). Wts \( D_U(p) \) discrete-convex.

Let \( y \in \text{Conv}D_U(p) \cap \mathbb{Z}^n \). Wts \( y \in D_U(p) = D_{u^1}(p) + D_{u^2}(p) \).
Proof of Sufficiency in Unimodularity Theorem

**Definition**

\[ L_\sigma := \langle \{y - x : x, y \in \sigma\} \rangle_\mathbb{R} \text{ where } \sigma \text{ is a demand complex cell.} \]

Suppose \( u^1, u^2 \) concave and \( p \in T_{u^1} \cap T_{u^2} \). Write \( \sigma^j, \sigma \) for \( \text{Conv } D_{u^j}(p), \text{Conv } D_U(p) \). Wts \( D_U(p) \) discrete-convex.

Let \( y \in \text{Conv } D_U(p) \cap \mathbb{Z}^n \). Wts \( y \in D_U(p) = D_{u^1}(p) + D_{u^2}(p) \).

Let \( x \in D_U(p) \). Prev slide: \( \exists z^j \in L_{\sigma^j} \cap \mathbb{Z}^n \) with \( y - x = z^1 + z^2 \).

But \( x \in D_U(p) \) so \( x = x^1 + x^2 \), with \( x^j \in D_{u^j}(p) \).

And \( y \in \sigma \) so \( y = y^1 + y^2 \) with \( y^j \in \sigma^j \) (not necessarily integer).
Definition

\[ L_\sigma := \langle \{ y - x : x, y \in \sigma \} \rangle \] where \( \sigma \) is a demand complex cell.

Suppose \( u^1, u^2 \) concave and \( p \in T_{u^1} \cap T_{u^2} \). Write \( \sigma^j, \sigma \) for \( \text{Conv} D_{u^j}(p), \text{Conv} D_U(p) \). Wts \( D_U(p) \) discrete-convex.

Let \( y \in \text{Conv} D_U(p) \cap \mathbb{Z}^n \). Wts \( y \in D_U(p) = D_{u^1}(p) + D_{u^2}(p) \).

Let \( x \in D_U(p) \). Prev slide: \( \exists z^j \in L_{\sigma^j} \cap \mathbb{Z}^n \) with \( y - x = z^1 + z^2 \).

But \( x \in D_U(p) \) so \( x = x^1 + x^2 \), with \( x^j \in D_{u^j}(p) \).

And \( y \in \sigma \) so \( y = y^1 + y^2 \) with \( y^j \in \sigma^j \) (not necessarily integer).

So \( z^1 + z^2 = (y^1 - x^1) + (y^2 - x^2) \).

Moreover, \( y^j - x^j \in L_{\sigma^j}, j = 1, 2 \). So \( z^j = x^j - y^j \). Since \( x^j, z^j \in \mathbb{Z}^n \) see \( y^j \in \mathbb{Z}^n \).

As \( u^j \) concave and \( y^j \in \sigma^j \cap \mathbb{Z}^n \), conclude \( y^j \in D_{u^j}(p) \).
The **strong substitute** vectors have at most one +1, at most one -1, otherwise 0s. Substitutes where trade-offs are 1-1.

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & -1
\end{pmatrix}
\]

- Unimodular set (classic result).
- Equilibrium always exists.
- Model of Kelso and Crawford (1982), Danilov et al. (2003), Milgrom and Strulovici (2009), Hatfield et al. (2013).
- The model of Sun and Yang (2006) is a basis change.
- All ‘product-mix auction’ bids express strong substitute preferences.
The **strong substitute** vectors have at most one +1, at most one -1, otherwise 0s. Substitutes where trade-offs are 1-1.

$$
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & 0 & -1 & -1
\end{pmatrix}
$$

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Beyond strong substitutes

\exists \text{ unimodular demand types, not a basis change from strong substitutes? If } n \leq 3, \text{ then no. If } n \geq 4, \text{ yes.}
Beyond strong substitutes

∃ unimodular demand types, not a basis change from strong substitutes? If \( n \leq 3 \), then no. If \( n \geq 4 \), yes. Let \( D \) be the columns of:

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

front-line workers \{ \\
manager \}

Interpretation:

- The first three goods (rows) represent front-line workers.
- The final good (row) is a manager.
- ‘Bundles’, i.e. teams, worth bidding for, are:
  - a worker on their own (not a manager on their own);
  - a worker and a manager;
  - two workers and a manager.
Beyond strong substitutes

∃ unimodular demand types, not a basis change from strong substitutes?
If $n \leq 3$, then no. If $n \geq 4$, yes. Let $\mathcal{D}$ be the columns of:

$$
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

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Interpret as coalitions: **model matching with transferable utility.**
Beyond strong substitutes

∃ unimodular demand types, not a basis change from strong substitutes? If \( n \leq 3 \), then no. If \( n \geq 4 \), yes. Let \( \mathcal{D} \) be the columns of:

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1
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\]

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Interpret as coalitions: model matching with transferable utility. Find unimodularity \( \Leftrightarrow \) core \( \not= \emptyset \).
\( \mathcal{D} \) is the basis change of the strong substitute vectors via the upper triangular matrix of 1s. Vectors with one block of consecutive 1s:

- People are ordered from 1 to \( n \).
- Subsets of consecutive people can form coalitions.
- Where there are ‘gaps’, no complementarity.

This can represent:

- small shops along a street considering a merger;
- seabed rights for oil / offshore wind.
Summary

Geometric analysis helps us understand

- **Individual Demand**
  - **See** an agent’s valuation and their trade-offs.
  - **Classify** valuations via ‘demand types’.
  - **Relate** one structure of trade-offs to another.

- **Aggregate Demand**
  - **Always** have competitive equilibrium iff ‘demand type’ is unimodular.
  - **Count intersections** to check for equilibrium in other cases.

- **Matching with transferable utility**
  - **Stability = equilibrium = unimodularity of set of putative coalitions.**
  - **New models** of multiparty stable matching: see ?.
We are revising this paper.

We would love to pay a graduate student (or anyone else!) to proof-read, especially the appendices, if anyone knows anyone who might be interested.

Please talk to me during the week or email e.c.baldwin@lse.ac.uk
V. Danilov and G. Koshevoy. Discrete convexity and unimodularity–I. *

V. Danilov, G. Koshevoy, and K. Murota. Discrete convexity and equilibria in economies with indivisible goods and money. *

V. Danilov, G. Koshevoy, and C. Lang. Gross substitution, discrete convexity, and submodularity. *

J. W. Hatfield, S. D. Kominers, A. Nichifor, M. Ostrovsky, and A. Westkamp. Stability and competitive equilibrium in trading networks. *

A. S. Kelso and V. P. Crawford. Job matching, coalition formation, and gross substitutes. *

D. Maclagan and B. Sturmfels. *Introduction to Tropical Geometry*, volume 161 of *

G. Mikhalkin. Decomposition into pairs-of-pants for complex algebraic hypersurfaces. *