# Simple Games and Power Indices

A **simple game** consists of

$N = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$, the players;

$v : \mathcal{P}(N) \to \{0, 1\}$, $\mathcal{P}(N)$ = power set of $N$, such that

- for all $S, T \in \mathcal{P}(N)$, $v(S) \leq v(S \cup T)$, equivalently, if $v(S) = 1$ and $S \subseteq T$, then $v(T) = 1$.
- $v(N) = 1$, $v(\emptyset) = 0$.

$v(S) = 1 \Rightarrow S$ is **winning**; otherwise **losing**.

A model for yes-no voting systems, e.g., legislatures, elections with two candidates.

Example: Weighted voting game. Assign real weights $w_i$ to voter $i$ ($w_i > 0$) and a quota $q$. $S$ winning iff $\sum_{i \in S} w_i \geq q$.

For a voter $k$ and a coalition $S$ containing $k$, we say that $k$ is **critical** in $S$ if $S$ is winning, and $S - \{k\}$ is losing. Let $C_{k,i}$ denote the number of coalitions of size $i$ for which $k$ is critical.

**power index**: $\Psi = \{\lambda_i\}_{i=1}^n$, $\lambda_i \in \mathbb{R}_{\geq 0}$, not all $\lambda_i = 0$. $\Psi(k) := \sum_{i=1}^n \lambda_i C_{k,i}$, the power of $k$. This yields an ordering of the voters.

Classic power indices: Banzhaf: $\lambda_i = 1/n$ for all $n$.
Shapley-Shubik: $\lambda_i = (i-1)!(n-i)!$.

Define a partial ordering on $N$: $k \gg m$ if $C_{k,i} \geq C_{m,i}$ for all $i$. If the partial ordering is complete, then every power index gives the same ordering of the voters.

Vague question: Properties of a simple game for which there are "few" orderings of the voters.

Specific question: Characterize games for which Banzhaf ordering = Shapley-Shubik ordering. Note: $n = 5$, always agree (ad-hoc argument). $n = 6$, don't always agree.