A Geometric Approach to Dominant Strategy Implementation

John A. Weymark

Vanderbilt University

References

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Notation and Basic Definitions

N: finite set of individuals

Ω: finite set of outcomes

T^i: i's characteristic type space for $i \in N$

$T^{-i} = \prod_{j \in N \setminus \{i\}} T^j$: the characteristic type space of individuals other than i

$T^i \times T^{-i}$: the characteristic type space

$(t^i, t^{-i}) \in T^i \times T^{-i}$: the characteristic type profile, which is private information

$v^i : \Omega \times T^i \to \mathbb{R}$: i's valuation function

A **direct mechanism** consists of

an **allocation function** $G\colon T^i \times T^{-i} \to \Omega$ and

a **payment function** $P \equiv (P^1, \dots P^n)\colon T^i \times T^{-i} \to \mathbb{R}^n$,

where $P^i\colon T^i \times T^{-i} \to \mathbb{R}^n$ is the **payment function for individual** i.

Given the other individuals' reported types $t^{-i} \in T^{-i}$, the **utility** of individual i with characteristic type $t^i \in T^i$ and reported type $s^i \in T^i$ is

$$v^i(G(s^i, t^{-i})|t^i) - P^i(s^i, t^{-i}).$$

An allocation function G is **dominant strategy implementable** if there exists a payment function P such that for all $i \in N$ and all $t^{-i} \in T^{-i}$,

$$v^i(G(t^i, t^{-i})|t^i) - P^i(t^i, t^{-i}) \geq v^i(G(s^i, t^{-i})|t^i) - P^i(s^i, t^{-i}), \quad \forall s^i, t^i \in T^i.$$

Given the allocation function G, for fixed $i \in N$ and $t^{-i} \in T^{-i}$, the *characteristic graph* $T_G(t^{-i})$ is the complete directed graph with nodes T^i and arc length

$$d(s^i, t^i | t^{-i}) = v^i(G(t^i, t^{-i})|t^i) - v^i(G(s^i, t^{-i})|t^i)$$

for the directed arc (s^i, t^i) from s^i to t^i.

Note that $d(s^i, t^i | t^{-i})$ is the increase in the valuation if the true characteristic type t is reported instead of the characteristic type s. This increase in valuation is not the increase in the utility because the payments have not been taken into account.

The Rochet–Rockafellar Theorem

For every integer $k \geq 2$, a k-**cycle** in the characteristic graph $T_G(t^{-i})$ is a sequence of arcs $(t_1, t_2), \ldots, (t_{k-1}, t_k), (t_k, t_1)$ whose **length** is defined to be the sum of the lengths of the arcs in the cycle, i.e., $d(t_1, t_2|t^{-i}) + \cdots + d(t_{k-1}, t_k|t^{-i}) + d(t_k, t_1|t^{-i})$.

Theorem 1 [Rockafellar (1970) – Rochet (1987)]

The allocation function $G\colon T^i \times T^{-i} \to \Omega$ is dominant strategy implementable if and only if for every $i \in N$, $t^{-i} \in T^{-i}$, and integer $k \geq 2$, all k-cycles in the characteristic graph $T_G(t^{-i})$ have nonnegative length.

$A = \{a_1, \ldots, a_m\}$ is the finite set of attainable outcomes given t^{-i}.

Note that m can depend on t^{-i}.

Let $R_a(t^{-i}) = \{t^i \in T^i \mid G(t^i, t^{-i}) = a\}$ be the set of characteristic types for i that induce outcome a with the allocation function G when the other individuals' types are given by t^{-i}.

By construction, $R_a(t^{-i})$ is nonempty for all $a \in A(t^{-i})$

Allocation Graphs

For the characteristic graph $T_G(t^{-i})$, the corresponding **allocation graph** $\Gamma_G(t^{-i})$ is the complete directed graph that has $A(t^{-i})$ as the set of nodes and $\ell(a, b|t^{-i})$ as the **length** of the directed arc from node a to node b, where for all distinct $a, b \in A(t^{-i})$,

$$\begin{aligned} \ell(a, b|t^{-i}) &= \inf_{t^i \in R_b(t^{-i})} \left[v^i(b|t^i) - v^i(a|t^i) \right] \\ &= \inf_{t^i \in R_b(t^{-i})} \left[v^i(G(t^i, t^{-i})|t^i) - v(a|t^i) \right]. \end{aligned}$$

For any integer $k \geq 2$, a k-**cycle** in the allocation graph $\Gamma_G(t^{-i})$ is a sequence of arcs $(a_1, a_2), \ldots, (a_{k-1}, a_k), (a_k, a_1)$ whose length is defined to be the sum of the lengths of its arcs in the cycle.

The Rockafellar–Rochet Theorem can be restated using allocation graphs by simply substituting the allocation graph $\Gamma_G(t^{-i})$ for the characteristic graph $T_G(t^{-i})$ in the statement of Theorem 1.

In order to analyze dominant strategy implementability, without loss of generality, we can consider a fixed individual $i \in N$ and fixed types $t^{-i} \in T^{-i}$ of the other individuals. Let $v = v^i$, $t = t^i$, $T = T^i$, and suppress the dependence of $A(t^{-i})$, $R_a(t^{-i})$, $d(s^i, t^i | t^{-i})$, and $\ell(a, b | t^{-i})$ on t^{-i}.

By fixing i and t^{-i}, (G, P) defines a *single person mechanism* (g, p) with allocation function $g\colon T \to A$ and payment function $p\colon T \to \mathbb{R}$ obtained by setting

$$g(t) = G(t, t^{-i}) \text{ and } p(t) = P^i(t, t^{-i}), \ \forall t \in T.$$

Note that g is surjective. The corresponding characteristic and allocation graphs are denoted by T_g and Γ_g, respectively.

For (g, p), the dominant strategy implementability condition is

$$v(g(t)|t) - p(t) \geq v(g(s)|t) - p(s) \quad \forall s, t \in T.$$

It follows that if g is dominant strategy implementable and $g(s) = g(t)$, then $p(s) = p(t)$ as well.

Theorem 2

The following conditions for the allocation function $g : T \to A$ are equivalent:

1. *g is dominant strategy implementable;*
2. *for every integer $k \geq 2$, all k-cycles in the characteristic graph T_g have nonnegative length;*
3. *for every integer $k \geq 2$, all k-cycles in the allocation graph Γ_g have nonnegative length.*

The 2-Cycle Nonnegativity Condition

An allocation function g satisfies the **characteristic graph 2-cycle nonnegativity condition** if

$$d(s,t) + d(t,s) \geq 0, \quad \forall s,t \in T, s \neq t$$

Note that this is equivalent to:

$$v(g(t)|t) - v(g(s)|t) \geq v(g(t)|s) - v(g(s)|s), \quad \forall s,t \in T, s \neq t.$$

That is, the increase in valuation obtained by replacing $g(s)$ with $g(t)$ is at least as large for t as for s. For this reason, the 2-cycle nonnegativity condition is also known as **weak monotonicity**.

An allocation function g satisfies the **allocation graph 2-cycle nonnegativity condition** if

$$\ell(a, b) + \ell(b, a) \geq 0, \quad \forall a, b \in A, a \neq b.$$

Theorem 3

An allocation function $g\colon T \to A$ satisfies the characteristic graph 2-cycle nonnegativity condition if and only if it satisfies the allocation graph 2-cycle nonnegativity condition.

The Saks–Yu Theorem

It follows straightforwardly from the incentive constraints that the 2-cycle nonnegativity condition is a necessary condition for an allocation function g to be dominant strategy implementable.

Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan, and Sen (2006) and Saks and Yu (2005) have identified restrictions on v under which the 2-cycle nonnegativity condition is sufficient for dominant strategy implementability. Our results build on Saks and Yu.

$\mathcal{V} = \{v \in \mathbb{R}^m | v = (v(a_1|t), \ldots, v(a_m|t)) \text{ for some } t \in T\}.$

$\mathcal{V}$ is i's **valuation type space** (given t^{-i}).

Each characteristic type $t \in T$ has associated with it a corresponding **valuation type** $v^t = (v^t_{a_1}, \ldots, v^t_{a_m}) \in \mathcal{V}$, where $v^t_a = v(a|t)$ for all $a \in A$.

Note that if characteristic types s and t have the same associated valuation type v, then there is no loss of generality in identifying them. Henceforth, we assume that if $s \neq t$, then $v^s \neq v^t$. With this assumption, there is a unique $t \in T$ associated with each $v \in \mathcal{V}$. Let t^v denote the characteristic type associated with v.

Proposition [Saks–Yu (2005)]

If $\mathcal{V}$ is convex, then the allocation function $g\colon T \to A$ is dominant strategy implementable if the 2-cycle nonnegativity condition is satisfied.

The following is the **Saks–Yu Theorem**.

Theorem 4 [Saks–Yu (2005)]

If $\mathcal{V}$ is convex, then the allocation function $g\colon T \to A$ is dominant strategy implementable if and only if the 2-cycle nonnegativity condition is satisfied.

Partitioning the Valuation Type Space

Recall that R_a is the set of characteristic types that the allocation function g maps into outcome a. The sets R_a for $a \in A$ induce a partition of the valuation type space $\mathcal{V}$. Our results are obtained by investigating the geometry of this partition.

For all $a, b \in A$ with $a \neq b$, the **difference set for** (a, b) is

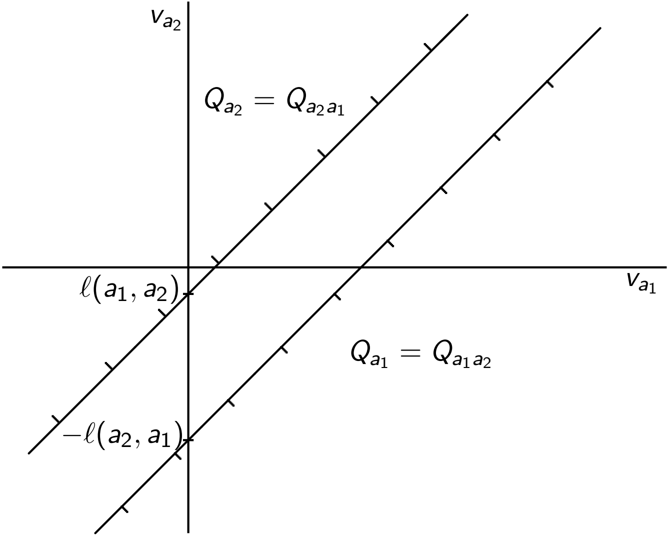
$$Q_{ab} = \{v \in \mathbb{R}^m | v_a - v_b \geq \ell(b, a)\}.$$

Q_{ab} is a closed halfspace in $\mathbb{R}^m$.

For all $a \in A$, let

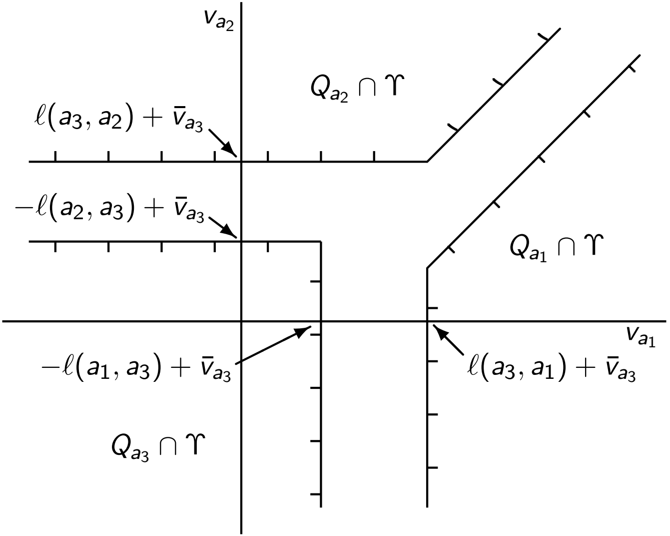
$$Q_a = \bigcap_{b \in A \setminus \{a\}} Q_{ab}.$$

Q_a is a closed convex polyhedron in $\mathbb{R}^m$.



$$Q_{a_1 a_2} = \{v \in \mathbb{R}^2 \mid v_{a_2} \leq -\ell(a_2, a_1) + v_{a_1}\}$$
$$Q_{a_2 a_1} = \{v \in \mathbb{R}^2 \mid v_{a_2} \geq \ell(a_1, a_2) + v_{a_1}\}.$$

2-cycle nonnegativity condition: $\ell(a_1, a_2) \geq -\ell(a_2, a_1)$



$$\Upsilon = \{v \in \mathbb{R}^3 \mid v_{a_3} = \bar{v}_{a_3}\}.$$

Theorem 5

For any allocation function $g\colon T \to A$ and any outcome $a \in A$, (i) for any characteristic type $t \in R_a$, the valuation type v^t is in $Q_a \cap \mathcal{V}$ and (ii) if g satisfies the 2-cycle nonnegativity condition, then for any valuation type $v \in Q_a^\circ \cap \mathcal{V}$, the characteristic type t^v is in R_a.

Theorem 6

If the allocation function $g \colon T \to A$ satisfies the 2-cycle nonnegativity condition, then for any characteristic type $t \in R_a$ and any valuation type $v' \in \mathcal{V}$ with $v' \geq v^t$ for which $v'_a > v^t_a$ and $v'_b = v^t_b$, the characteristic type $t^{v'}$ is not in R_b.

Theorem 6 is a monotonicity result. Suppose that outcome *a* is chosen. If the valuation type increases in the value of outcome *a* and does not decrease in the valuation of any other outcome, then with the new valuation type, no outcome can be chosen whose valuation has not changed.

Zero Length Cycles

The valuation type space $\mathcal{V}$ is a **full-dimensional convex product space** if

$$\mathcal{V} = \times_{a \in A} \langle L_a, U_a \rangle,$$

where for all $a \in A$, $\langle L_a, U_a \rangle$ is any type of interval of $\mathbb{R}$ with endpoints L_a and U_a for which $L_a < U_a$.

Interiority assumption: $Q_a^\circ \cap \mathcal{V}^\circ \neq \varnothing$ for all $a \in A$.

Theorem 7

Suppose that $|A| \geq 2$. If (i) the allocation function $g\colon T \to A$ satisfies the 2-cycle nonnegativity condition, (ii) the valuation type space $\mathcal{V}$ is a full-dimensional convex product space, and (iii) $Q_a^\circ \cap \mathcal{V}^\circ \neq \varnothing$ for all $a \in A$, then for every integer $k \geq 2$, all k-cycles in the allocation graph Γ_g have zero length.

Theorem 7 is established by a series of lemmas.

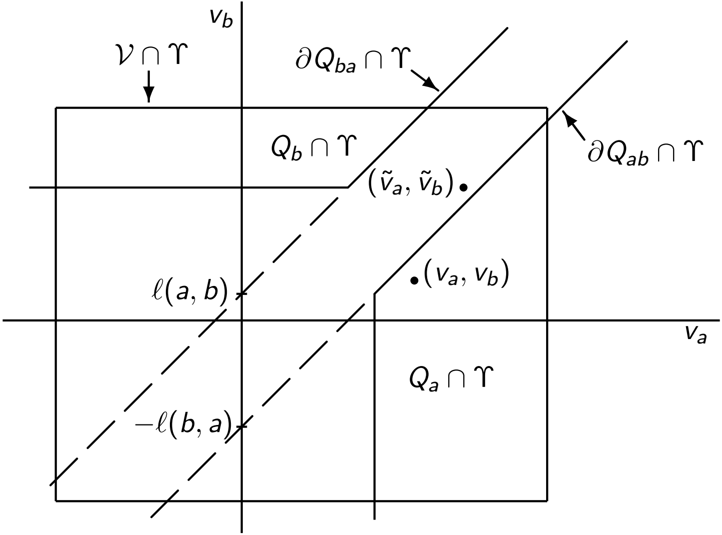
Lemma 1

Under the assumptions of Theorem 7, any 2-cycle in the allocation graph Γ_g has zero length.

For the special case in which $\mathcal{V}$ is all of $\mathbb{R}^m$, Lemma 1 has been established in Lavi, Mu'alem, and Nisan (2009).

If there are only two outcomes (i.e., if $\mathcal{V} \subseteq \mathbb{R}^2$), then the conclusion of Lemma 1 holds if the allocation rule g satisfies the 2-cycle nonnegativity condition and the value type space $\mathcal{V}$ is a convex.

When there are three or more outcomes, the conclusion of Lemma 1 need not hold if the interiority assumption is not satisfied.


v_b
$\mathcal{V} \cap \Upsilon$
$\partial Q_{ba} \cap \Upsilon$
$Q_b \cap \Upsilon$
$\partial Q_{ab} \cap \Upsilon$
$(\tilde{v}_a, \tilde{v}_b)$
(v_a, v_b)
$\ell(a, b)$
v_a
$Q_a \cap \Upsilon$
$-\ell(b, a)$

Lemma 2

If all 2-cycles in the allocation graph Γ_g have zero length and all 3-cycles in Γ_g have nonnegative length, then for every integer $k \geq 2$, any k-cycle in Γ_g has zero length.

Consider any 3-cycle $(a_1, a_2), (a_2, a_3), (a_3, a_1)$. Because all 3-cycles have nonnegative length,

$$\ell(a_1, a_2) + \ell(a_2, a_3) + \ell(a_3, a_1) \geq 0.$$

Because all 2-cycles have zero length, this inequality is equivalent to

$$-\ell(a_2, a_1) - \ell(a_3, a_2) - \ell(a_1, a_3) \geq 0,$$

or, equivalently,

$$\ell(a_1, a_3) + \ell(a_3, a_2) + \ell(a_2, a_1) \leq 0.$$

Because all 3-cycles have nonnegative length, the last inequality implies that the 3-cycle $(a_1, a_3), (a_3, a_2), (a_2, a_1)$ must have zero length, which implies that the original 3-cycle $(a_1, a_2), (a_2, a_3), (a_3, a_1)$ must have zero length.

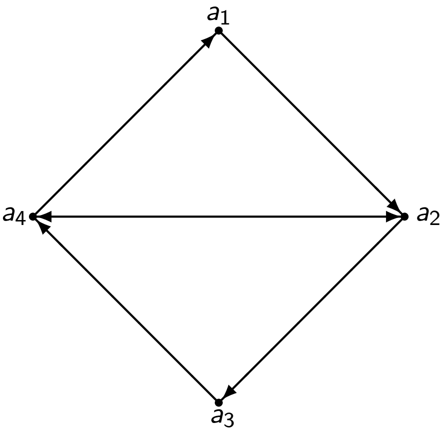
Consider any 4-cycle $(a_1, a_2), (a_2, a_3), (a_3, a_4), (a_4, a_1)$. Because all 2-cycles have zero length, the length of this 4-cycle is equal to the sum of the lengths of the following 3-cycles:

$(a_1, a_2), (a_2, a_4), (a_4, a_1)$

$(a_2, a_3), (a_3, a_4), (a_4, a_2)$

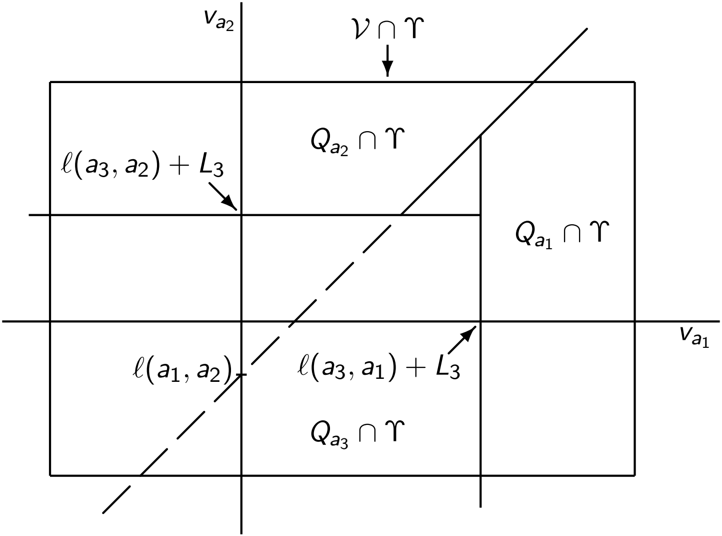
both of which have length zero.

Induction is used to prove the lemma for larger values of k.

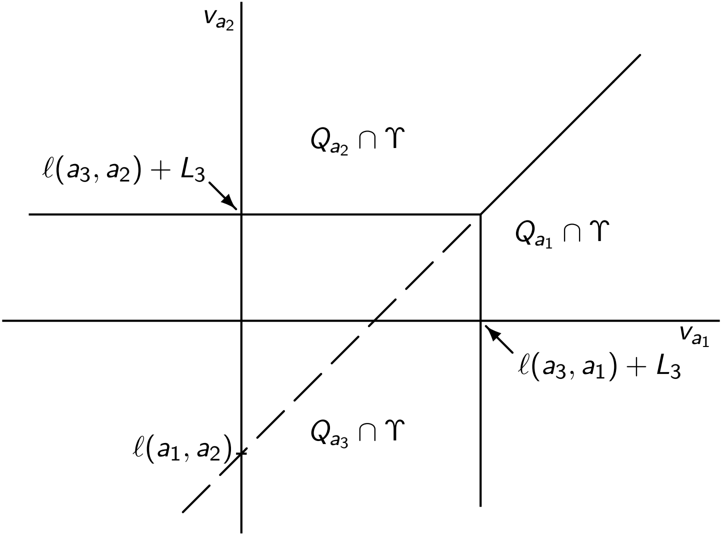

a_1
a_4
a_2
a_3

Lemma 3

Under the assumptions of Theorem 7, any 3-cycle in the allocation graph Γ_g *has nonnegative length.*



$$\Upsilon = \{v \in \mathbb{R}^3 \mid v_3 = L_3\}.$$



$$\ell(a_3, a_2) + L_3 = \ell(a_1, a_2) + \ell(a_3, a_1) + L_3 \leftrightarrow$$

$$\ell(a_1, a_2) + \ell(a_2, a_3) + \ell(a_3, a_1) = 0.$$

Theorem 8

If (a) $|A| = 1$ or (b) $|A| \geq 2$, the valuation type space $\mathcal{V}$ is a full-dimensional convex product space, and $Q_a^\circ \cap \mathcal{V}^\circ \neq \varnothing$ for all $a \in A$, then the following conditions for the allocation function $g : T \to A$ are equivalent:

1. *g is dominant strategy implementable;*
2. *for every integer $k \geq 2$, all k-cycles in the allocation graph Γ_g have nonnegative length;*
3. *for every integer $k \geq 2$, all k-cycles in the characteristic graph T_g have nonnegative length;*
4. *all 2-cycles in the allocation graph Γ_g have nonnegative length;*
5. *all 2-cycles in the characteristic graph T_g have nonnegative length;*
6. *all 2-cycles in the allocation graph Γ_g have zero length.*

Affine Maximizers

For all i, let $\ell(a_i, a_i) = 0$.

Theorem 9

If all 2-cycles and all 3-cycles in the allocation graph Γ_g have zero length, then

$$g(t) \in \arg\max_{a_i \in A} \left\{ v^t_{a_i} - \frac{1}{m} \sum_{j=1}^{m} \ell(a_j, a_i) \right\}, \quad \forall t \in T.$$

To illustrate that proof strategy, suppose that $m = 3$ and that a_1 is in the argmax. Then,

$$v^t_{a_1} - \frac{1}{3}[\ell(a_2, a_1) + \ell(a_3, a_1)] \geq v^t_{a_2} - \frac{1}{3}[\ell(a_1, a_2) + \ell(a_3, a_2)].$$

Because 2-cycles have zero length,

$$v^t_{a_1} - v^t_{a_2} - \frac{2}{3}\ell(a_2, a_1) \geq \frac{1}{3}[\ell(a_3, a_1) - \ell(a_3, a_2)].$$

So,

$$v^t_{a_1} - v^t_{a_2} - \ell(a_2, a_1) \geq \frac{1}{3}[\ell(a_3, a_1) + \ell(a_2, a_3) + \ell(a_1, a_2)].$$

Because 3-cycles have zero length,

$$v^t_{a_1} - v^t_{a_2} \geq \ell(a_2, a_1).$$

For all i, let

$$k_i = -\frac{1}{m}\sum_{j=1}^{m} \ell(a_j, a_i)$$

Let $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$.

Define the tropical polynomial $F\colon \mathcal{V} \to \mathbb{R}$ by setting

$$F(v) = k_1 \odot v_{a_1} \oplus k_2 \odot v_{a_2} \oplus \cdots \oplus k_m \odot v_{a_m}, \quad \forall v \in \mathcal{V}.$$

We have shown that $g(t)$ is one of the outcomes in $A = \{a_1, \ldots, a_m\}$ that maximizes $F(v^t)$ when all of the 2-cycles and 3-cycles have zero length.

The tropical hypersurface of F is $\cup_{a_i \in A}\, \partial Q_{a_i}$.

If we consider all of the outcomes in Ω, individual i's *valuation type space* is

$$\mathcal{V}^i = \{(v^i(a_1|t^i), \dots, v^i(a_M|t^i)) \in \mathbb{R}^M \mid t^i \in T^i\}.$$

where $M = |\Omega|$.

$\mathcal{V}^i$ is *unrestricted* if $\mathcal{V}^i = \mathbb{R}^M$.

The allocation function $G\colon T^i \times T^{-i} \to \Omega$ is *nonimposed* if $G(T^i \times T^{-i}) = \Omega$

G is an *affine maximizer* if there exist n nonnegative numbers $w_1, \ldots, w_n$, not all of them equal to zero, and M numbers K_a, $a \in \Omega$, such that

$$G(t^i, t^{-i}) \in \arg\max_{a \in \Omega} \left[\sum_{j=1}^{n} w_j v^j(a|t^j) + K_a \right], \quad \forall (t^i, t^{-i}) \in T^i \times T^{-i}.$$

The affine maximizer G is *unresponsive to irrelevant agents* if for all $i \in N$ for which $w_i = 0$, $G(s^i, t^{-i}) = G(t^i, t^{-i})$ for all $s^i, t^i \in T^i$ and all $t^{-i} \in T^{-i}$.

Theorem 10

(a) If an allocation function $G: T^i \times T^{-i} \to \Omega$ *is an affine maximizer that is unresponsive to irrelevant agents, then* G *is dominant strategy implementable.*

(b) Suppose that there are at least three outcomes in Ω, $\mathcal{V}^i$ is unrestricted for all $i \in N$, and $G: T^i \times T^{-i} \to \Omega$ is a nonimposed allocation function. If G is dominant strategy implementable, then G is an affine maximizer.

Part (a) is due to Mishra and Sen (2012). Part (b) is due to Roberts (1979).

Example

There is one unit of an indivisible good to be allocated to one of two individuals. Possession of the good creates a negative externality for the other individual.

a (resp. b) is the outcome in which individual 1 (resp. 2) gets the good.

$T^1 = \mathbb{R}_+ \times \mathbb{R}_-$ and $T^2 = \mathbb{R}_- \times \mathbb{R}_+$

$v^1(a|t^1) = t^1_a$ and $v^1(b|t^1) = t^1_b$ for all $t^1 \in T^1$.

$v^2(b|t^2) = t^2_b$ and $v^2(a|t^2) = t^2_a$ for all $t^2 \in T^2$.

A Vickrey auction has the following allocation and payment functions:

$$G(t^1, t^2) = \begin{cases} a & \text{if } t_a^1 - t_b^1 \geq t_b^2 - t_a^2 \\ b & \text{if } t_a^1 - t_b^1 < t_b^2 - t_a^2 \end{cases}$$

and payment function $P\colon T^1 \times T^2 \to \mathbb{R}^2$ is

$$P(t^1, t^2) = \begin{cases} (t_b^2 - t_a^2, 0) & \text{if } t_a^1 - t_b^1 \geq t_b^2 - t_a^2 \\ (0, t_a^1 - t_b^1) & \text{if } t_a^1 - t_b^1 < t_b^2 - t_a^2. \end{cases}$$

Each person has an adjusted value for the good given by $t_a^1 - t_b^1$ for person 1 and $t_b^2 - t_a^2$ for person 2. Individuals bid their adjusted values and the good is awarded to the highest bidder (with a tie broken in favour of individual 1) with the winner paying the second-highest bid (in this case, the other person's bid) and the loser paying nothing. Note that G chooses the outcome that maximizes the sum of the valuations.

For any $t^2 \in T^2$, $A(t^2) = \{a, b\}$. We have

$$\ell(a, b|t^2) = \inf_{t^1 \in R_b(t^2)} [v(b|t^1) - v(a|t^1)]$$
$$= \inf_{t_b^1 - t_a^1 > -[t_b^2 - t_a^2]} [t_b^1 - t_a^1] = -[t_b^2 - t_a^2]$$

because $R_b(t^2) = \{t^1 \in T^1 \mid t_a^1 - t_b^1 < t_b^2 - t_a^2\}$ and we have

$$\ell(b, a|t^2) = \inf_{t^1 \in R_a(t^2)} [v(a|t^1) - v(b|t^1)] = \inf_{t_a^1 - t_b^1 \geq t_b^2 - t_a^2} [t_a^1 - t_b^1] = t_b^2 - t_a^2$$

because $R_a(t^2) = \{t^1 \in T^1 \mid t_a^1 - t_b^1 \geq t_b^2 - t_a^2\}$. Thus, the only 2-cycle in the allocation graph $\Gamma_G(t^2)$ has zero length. Similarly, for any $t^1 \in T^1$, $A(t^1) = \{a, b\}$ and the only 2-cycle in $\Gamma_G(t^1)$ has zero length.