Computing Linear Systems on Metric Graphs

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Background

- Hasse, Musiker and Yu (’12): Linear systems on tropical curves.
Outline

- Introduction
- $|D|$ as a cell complex
- $R(D)$ as a tropical semi-module
- Examples of canonical linear systems (motivated from tropical curves)
- Open problems
Metric graphs

Definition

A metric graph $\Gamma$ is a connected undirected finite graph whose edges have lengths. It is determined by its graph-theoretic type (called skeleton) and the lengths of its edges (called metric).
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Remark

We also denote by $\Gamma$ the set of all vertices and interior points of the metric graph, if without any confusion.
Definition

A divisor \( D \) on \( \Gamma \) is a formal finite \( \mathbb{Z} \)-linear combination
\[
D = \sum_{x \in \Gamma} D(x) \cdot x \text{ of points of } \Gamma.
\]
The divisor is effective if \( D(x) \geq 0 \) for all \( x \in \Gamma \).
The degree of a divisor \( D \) is \( \sum_{x \in \Gamma} D(x) \).
The support of a divisor \( D \) on \( \Gamma \) is the set \( \{ x \in \Gamma | D(x) \neq 0 \} \), denoted as \( \text{supp}(D) \).
Rational functions

Definition

A tropical rational function \( f \) on \( \Gamma \) is a continuous function \( f : \Gamma \to \mathbb{R} \) that is piecewise-linear on each edge with finitely many pieces and integral slopes. The order \( \text{ord}_x(f) \) of \( f \) at a point \( x \in \Gamma \) is the sum of outgoing slopes at \( x \) towards all directions. Note that if \( x \) is an interior point of a linear piece of \( f \), then \( \text{ord}_x(f) = 0 \). The principal divisor associated to \( f \) is

\[
(f) = \sum_{x \in \Gamma} \text{ord}_x(f) \cdot x.
\]
**Definition**

For any divisor $D$ on $\Gamma$, let $R(D)$ be the set of all rational functions $f$ on $\Gamma$ such that the divisor $D + (f)$ is effective, and $|D| = \{D + (f) | f \in R(D)\}$, the linear system of $D$. 

**Remark**

Let $\mathbb{1}$ be the set of constant functions on $\Gamma$. There is a bijection between the rational functions in $R(D) / \mathbb{1}$ and the effective divisors in $|D|$. However, they have different combinatorial structures and we explore them in this work.
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Remark

Let $1$ be the set of constant functions on $\Gamma$. There is a bijection between the rational functions in $R(D)/1$ and the effective divisors in $|D|$. However, they have different combinatorial structures and we explore them in this work.
Example $C_4$

Let $\Gamma = (V, E)$ be a metric graph with skeleton $C_4$ and edges with equal lengths. Let $D = \sum_{v \in V} 2 \cdot v$. The following figures show a rational function $f \in R(D)$ on $\Gamma$ and the corresponding effective divisor $D + (f)$. 
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**Figure: $\Gamma$**

**Figure: The divisor $D$**
Figure: rational function $f$ (blue) on the metric graph $\Gamma$ (black)
Background and definitions
The cell complex $|D|$
Tropical convex set $R(D)$
Examples of canonical linear systems
Open Problems

Figure: The effective divisor $D + (f)$
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From now the divisor $D$ is always vertex-supported, unless specified. However, many divisors in $|D|$ are not vertex-supported.
Cells

We identify each open edge $e \in E$ with the interval $(0, M_e)$. Then each open cell of $|D|$ is characterized by the following data:

- a nonnegative integer $d_v$ for each $v \in V$;
- an ordered partition $d_e = \sum_{i=1}^{r_e} d_{e,i}$ of positive integers for some $e \in E$;
- an integer $m_e$ for each $e \in E$. 

Then a divisor $L$ belongs to this cell if and only if $L(v) = d_v$ for each $v \in V$;

For each $e \in E$, $L$ on $e$ is either expressed as $\sum_{i=1}^{r_e} d_{e,i} \cdot x_i$, where $0 < x_1 < x_2 < \ldots < x_{r_e} < M_e$, or zero.

Suppose $f \in R(D)$ such that $L = D + (f)$, then the outgoing slope of $f$ at the point $0$ is $m_e$ for each $e \in E$. 

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- Suppose $f \in R(D)$ such that $L = D + (f)$, then the outgoing slope of $f$ at the point 0 is $m_e$ for each $e \in E$. 

Dimension of a cell

Given the data of a cell, we would like to know the dimension of the cell.
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**Proposition**

(Haase-Musiker-Yu, 2012) Let $D$ be a vertex-supported divisor on a metric graph $\Gamma$ and $V$ is the set of vertices in $\Gamma$. Let $C$ be a cell in $|D|$ and divisor $L$ is a representative of $C$. Let $I_L = \text{supp}(L) - V$. Then $\dim C$ is one less than the number of connected components in the graph $\Gamma - I_L$. 
Anchor cells

In order to find all cells in $|D|$, we introduce the *anchor cells*, which serve as the landmarks in $|D|$.

**Definition**

A divisor $L$ on $\Gamma$ is an anchor divisor if for each edge of $\Gamma$ there is at most one its interior point $x$ with $L(x) > 0$. A cell $C$ in $|D|$ is an anchor cell if all divisors in $C$ are anchor divisors.
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**Lemma**

If $f \in R(D)$ and $D + (f)$ is an anchor divisor, then $f$ has at most 2 linear pieces on each edge of $\Gamma$. 
Corollary

Let $D$ be a vertex-supported divisor on a metric graph $\Gamma$. If $C$ is an anchor cell and it is represented by a divisor $D + (f)$, then $C$ is uniquely determined by the outgoing slopes of $f$ at all vertices of $\Gamma$. 
Corollary

Let $D$ be a vertex-supported divisor on a metric graph $\Gamma$. If $C$ is an anchor cell and it is represented by a divisor $D + (f)$, then $C$ is uniquely determined by the outgoing slopes of $f$ at all vertices of $\Gamma$.

Lemma (Haase-Musiker-Yu, 2012)

Let $D$ be a divisor on a metric graph $\Gamma$ and $f \in R(D)$. Then the slopes of all linear pieces of $f$ are between $-\deg(D)$ and $\deg(D)$.
With the two results above we can prove the following proposition.

**Proposition**

Let $D$ be a vertex-supported divisor on a metric graph $\Gamma$. Then there are finitely many anchor cells in $|D|$.
Finiteness

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Let $D$ be a vertex-supported divisor on a metric graph $\Gamma$. Then there are finitely many anchor cells in $|D|$.

**Proof.**

Every anchor cell $C$ in $|D|$ is represented by a divisor $D + (f)$, where $f \in R(D)/1$. Since $C$ is uniquely determined by the outgoing slopes of $f$ at all vertices of $\Gamma$, there are $2|E|$ slopes to assign. In addition, all these slopes belong to the interval $[-d, d]$ where $d = \deg(D)$ and they are integers. So there are finitely many choices for each of them, and thus for all of them.
Algorithm

Our algorithm to find all cells of $|D|$ consists of two steps:

1. Find all anchor cells of $|D|$ using linear programming algorithm.

2. Associate every cell to a unique anchor cell. For each anchor cell, find all cells associated to it.
Given $\Gamma$ and $D$, there are two approaches using linear programming to find all anchor cells in $|D|$.
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LP approach - search among all configuration of $D + (f)$

Note that if $D + (f)$ is an anchor divisor in $|D|$, then the support of this divisor has at most one intersection point with each edge of $\Gamma$. Then this divisor corresponds to a configuration of $\deg(D)$ unordered chips into $|V| + |E|$ bins (could be empty). There are $\binom{|V| + |E| + \deg(D) - 1}{\deg(D) - 1}$ such configurations.
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For each configuration, we introduce the following $|V| + 2|E|$ variables: the value of $f$ at each vertex in $V$ and the outgoing slopes of $f$ at both endpoints of all edges in $E$. Then we have a system of linear equations and inequalities. There is a corresponding anchor cell if and only if the system has a solution where all slopes are integers.
Association

For any cell $C$ of $|D|$, choose a divisor $L$ in it to represent. If $L$ is already anchor, then $C$ is associated to itself; otherwise for every edge that has more than one interior point with chip in $L$, we can apply a local chip-firing to combine all these chips to the same point in a unique way. After applying these chip-firings, we end up with an anchor divisor $L'$ in an anchor cell $C'$, and the $d_v$ and $m_e$ are unchanged. We associate $C$ to $C'$. 
Example $C_4$ continued

Example

The left divisor represents a 3-dimensional cell and it is associated to a 2-dimensional anchor cell, which is represented by the anchor divisor on the right.

Figure: The left divisor is associated to the right one
The following theorem provides a combinatorial formula to compute the $f$-vector $(f_0, f_1, \ldots, f_d)$ of $|D|$ given all of its anchor cells. Here $f_i$ is the number of $i$-dimensional cells in $|D|$.

**Theorem (Lin, 2016)**

Let $D$ be a vertex-supported divisor on a metric graph $\Gamma$. If $C_1, C_2, \ldots, C_m$ are all anchor cells in $|D|$, and for $1 \leq i \leq m$, $C_i$ is $d_i$-dimensional and is represented by the divisor $A_i$ and $c_i = \sum_{x \in \Gamma-V} A_i(x)$, $e_i$ is the number of edges in $\Gamma$ that contain an interior point $y$ with $A_i(y) > 0$, and the $f$-vector of $|D|$ is $(f_0, f_1, \ldots, f_d)$, then

$$
\sum_{k=0}^d f_k x^k = \sum_{i=1}^m x^{d_i} (1 + x)^{c_i - e_i}.
$$
Proof.

If there are $k$ chips at an interior point of an edge for an anchor divisor $A_i$, then there are $2^{k-1}$ ways to partition the $k$ chips along the edge, and the contribution to the dimension is one less than the number of parts.
Sketch of proof

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Remark

The right divisor in the previous example associates $2^2 + 2^2 + 3^3 - 3 = 16$ cells. Among them there are 1, 4, 6, 4, 1 cells of dimension 2, 3, 4, 5, 6 respectively.
We can define the sum of rational functions in $R(D)$ in a natural way $(f + g)(x) = f(x) + g(x)$ for $x \in \Gamma$. Then the new function $f + g$ is still a tropical rational function on $\Gamma$, but it may not belong to $R(D)$. 
Tropical convexity

**Definition**

For rational functions $f, g$ on $\Gamma$, we can define their (tropical) sum as

$$(f + g)(x) = \max(f(x), g(x))$$

for all $x \in \Gamma$. Similarly the scalar multiplication of $f$ with a scalar $c$ is the sum of $f$ with the constant function that always takes value $c$. 

**Lemma** (Haase-Musiker-Yu, 2012) Let $D$ be any divisor on a metric graph $\Gamma$. The space $R(D)$ is a tropical semi-module (convex set), i.e. it is closed under tropical addition and tropical scalar multiplication.
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Extremal generators

Definition

A function $f \in R(D)$ is called extremal if for any $g_1, g_2 \in R(D)$, $f = g_1 \oplus g_2 \Rightarrow f = g_1$ or $f = g_2$. 
Extremal generators

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\]

**Proposition (Haase-Musiker-Yu, 2012)**

The tropical semi-module \( R(D) \) is generated by the extremal generators.
Chip-firing

Definition

For a metric graph $\Gamma$, its subgraph is a compact subset with a finite number of components. Fix an effective divisor $L$ on $\Gamma$. We say a subgraph $\Gamma'$ of $\Gamma$ can fire for $L$ if for each boundary point $x$ of $\Gamma' \cap \overline{\Gamma - \Gamma'}$ the number of edges pointing out of $\Gamma'$ is no greater than $L(x)$. 

Remark

If $\Gamma'$ can fire for $L$, then there exists a rational function $f \in R(L)$ such that the divisor $L + (f)$ is obtained from $L$ by moving one chip along each edge pointing out of $\Gamma'$ by a small distance.
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We use the procedure of *chip-firing* to get a simple criterion of extremal generators.
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**Lemma (Haase-Musiker-Yu, 2012)**

Let $D$ be any divisor on a metric graph $\Gamma$. Then $f \in R(D)$ is extremal if and only if there are not two proper subgraphs $\Gamma_1$ and $\Gamma_2$ of $\Gamma$ such that they cover $\Gamma$ and both can fire on $D + (f)$. 
A non-extremal function

Let $\Gamma$ has skeleton $K_{3,3}$ and $D$ be the canonical divisor $K = \sum_{v \in V} v$.

**Figure:** A divisor $K + (f)$ with non-extremal $f \in R(K)$ and the two subgraphs (red) that can fire. The corresponding rational functions take value 1 on the red parts and 0 on the black parts and are linear with slope 1 from red parts to black parts.
Proposition (Haase-Musiker-Yu, 2012)

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**Remark**

Then we can search for extremal generators among the vertices of $|D|$ by the criterion of chip-firing.
In this section we fix the skeleton of a metric graph $\Gamma$ and let the metric vary in $\mathbb{R}_+^{E}$. We also choose $D$ as the canonical divisor

$$K = \sum_{v \in V} (d(v) - 2) \cdot v.$$ 

It turns out that different metrics give different cell complex structures.
We fix $K_4$ as the skeleton of $\Gamma$ and $D = K$. The $f$-vector is $(f_0, f_1, \ldots, f_d)$. 
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<table>
<thead>
<tr>
<th>Metric</th>
<th>Anchor Cells</th>
<th>Extremal Generators</th>
<th>$f$-vector</th>
<th>$C_3$</th>
<th>$C_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 1, 1, 1, 1, 1)$</td>
<td>30</td>
<td>7</td>
<td>$(14, 28, 15)$</td>
<td>12</td>
<td>3</td>
</tr>
<tr>
<td>$(1, 1, 2, 2, 1, 1)$</td>
<td>42</td>
<td>11</td>
<td>$(26, 52, 31, 4)$</td>
<td>24</td>
<td>7</td>
</tr>
<tr>
<td>$(2, 2, 2, 2, 2, 3)$</td>
<td>36</td>
<td>9</td>
<td>$(20, 40, 23, 2)$</td>
<td>18</td>
<td>5</td>
</tr>
<tr>
<td>$(4, 9, 7, 8, 6, 10)$</td>
<td>50</td>
<td>15</td>
<td>$(34, 60, 27)$</td>
<td>12</td>
<td>15</td>
</tr>
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**Table:** Structure of $|K|$ and $R_K$ for different metrics on $K_4$
We fix $K_{3,3}$ as the skeleton of $\Gamma$ and $D = K$. The $f$-vector is $(f_0, f_1, \ldots, f_d)$. 
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<td>3 91 96</td>
<td>94 4 92</td>
<td>93 95 5</td>
<td>730</td>
</tr>
<tr>
<td>2 1 1</td>
<td>1 2 1</td>
<td>1 1 2</td>
<td>460</td>
</tr>
<tr>
<td>All-equal</td>
<td>370</td>
<td>33</td>
<td>(130, 483, 630, 348, 81, 9)</td>
</tr>
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**Table:** Structure of $R_{M,k}$ and $|K|$ for different metrics on $K_{3,3}$
In combinatorics

- Given all the cells in $|D|$, find the face lattice of $|D|$.
- Fix the skeleton of $\Gamma$ and $D$, find non-trivial upper and lower bounds of the number of anchor cells (or cells, vertices) in $|D|$ and of the number of extremal generators in $R(D)$.
- Find all the possible combinatorial types of $d$-dimensional polytopes that can occur as cells in a linear system $|D|$. This is already very interesting when $d = 3$. 
In tropical geometry

- Analyze the canonical embedding of $R(D)$ into $\mathbb{TP}^{m-1}$, where $m$ is the number of extremal generators in $R(D)$.
- Find the linear dependence of the extremal generators in $R(D)$ (an analogue of Petri’s Theorem).
Some references

Thanks!


