Tropical Geometry in Economics

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Auction

An auction is a mechanism to sell goods.

inputs ▶ supply and reserve price from the sellers
        ▶ bids from the buyers

outputs ▶ amount of goods each buyer gets
           ▶ how much each buyer pays

Example
There is a single undivisible item for sale. There is a single winner, who is the higher bidder. Different types of auctions:

▶ descending vs. ascending
  (price = highest or the second highest bid)

▶ open vs. sealed bids,

▶ single- vs. multiple- rounds

Revenue Equivalence Theorem [Vickrey 1961] (Nobel Prize 1996)
All types of auctions give the same expected revenue under some general assumptions that bidders act independently and rationally.
Multiple types of goods

“How should goods that both sellers and buyers view as imperfect substitutes be sold, especially when multi-round auctions are impractical?”

—Paul Klemperer

Possible objectives:

- fair pricing (each winner pays the same per-unit prices)
- optimizes surplus (= bid – price) for each buyer
- supply = demand
- optimizes revenue for the sellers
- truthfulness of bidders
- prevents colluding among bidders
Product-mix auctions: motivation

- The **product-mix auction** is a single-round sealed bid auction invented by Paul Klemperer during the 2007 financial crisis to help the Bank of England auction off loans to banks.

- different goods = loans against different collaterals.

- per-unit price = interest rate.

Product-mix auctions: the set-up

- $n = \#$ types of indivisible goods.
- A bundle of goods is a point in $\mathbb{Z}^n$. By allowing negative quantities, we treat buyers and sellers the same way.
- At price $p \in \mathbb{R}^n$, the agent pays $p \cdot b$ for a bundle $b \in \mathbb{Z}^n$.
- The set of available bundles for each agent (buyer/seller) is a finite set $A \subset \mathbb{Z}^n$.
- Each agent has a valuation $u : A \rightarrow \mathbb{R}$, which gives a utility / surplus / profit function on the price space $\mathbb{R}^n$, given by $p \mapsto \max_{b \in A} \{u(b) - p \cdot b\}$.
- The demand set at price $p$ is $D_u(p) = \arg\max_{x \in A} \{u(x) - p \cdot x\}$, the set of bundles that maximizes utility at price $p$. 
For each agent, fix the set $A$ of available bundles and a valuation $u$.

- The price space $\mathbb{R}^n$ is partitioned into convex polyhedral cells according to the demand set $D_p(u)$.

- The boundary between maximal cells is called the **locus of indifference prices** (LIP) $\mathcal{L}_u = \{ p \in \mathbb{R}^n : \#D_u(p) > 1 \}$.

- It is “dual” to the **demand complex**
  \[
  \{ \text{conv}(D_u(p)) : p \in \mathbb{R}^n \},
  \]
  a polyhedral complex in the quantity space. Each cell corresponds to a set of bundles that are demanded at the same price.

- The demand complex is a **regular subdivision** of $A$ obtained by lifing the each point $a \in A$ to height $u(a)$ and projecting back down the upper convex hull.

- Edge directions of the demand complex
  $= \text{vectors perpendicular to wall of LIP}$

  They are called **demand types**, which generalizes the notions of substitutes and complements.
Aggregation

For multiple agents, \( j = 1, \ldots, m \):

- Set of available bundles \( A_j \subset \mathbb{Z}^n \) for each \( j \).
- Valuation \( w^j : A_j \rightarrow \mathbb{R} \) for each \( j \).
- The aggregate demand is the Minkowski sum \( D_p(U) = D_p(u^1) + \cdots + D_p(u^m) \).
- Aggregate demand complex is a mixed subdivision of \( A_1 + A_2 + \cdots + A_m \).
Competitive equilibrium (CE) exists at supply \( a \in \mathbb{Z}^n \) if \( a \) is in the aggregate demand set for some price \( p \). That is, there exist

- a choice of per-unit prices \([p \in \mathbb{R}^n]\)
- an assignment of product bundle for each agent, \([b_1, \ldots, b_m \in \mathbb{Z}^n]\)

such that

- the assigned bundle maximizes profit (surplus) for each agent at the chosen price, \([b_i \in D_p(u^i) \text{ for all } i]\)
- the market clears, supply = aggregate demand, \([a = \sum_{i=1}^{m} b_i]\)

When does competitive equilibrium exist?
How do we find the equilibrium prices and the distribution of bundles?
Tropical mathematics

- Consider the set $\mathbb{R}$ with two operations: $\oplus = \max$, $\odot = +$.
- For example,
  \[(3 \odot x \odot y \odot y) \oplus (5 \odot x \odot x \odot y) \oplus 0 = \max\{3 + x + 2y, 5 + 2x + y, 0\}\]
as functions.
- Tropical polynomials = piecewise linear convex functions with rational slopes.
- The tropical hypersurface $\mathcal{T}(F)$ of a tropical polynomial $F$ is its corner locus, where the max is attained at least twice.
- $\mathcal{T}(F \odot G) = \mathcal{T}(F) \cup \mathcal{T}(G)$.
  (Compare with: $\text{ZeroSet}(f \cdot g) = \text{ZeroSet}(f) \cup \text{ZeroSet}(g)$ for ordinary polynomials $f, g$ over a field.)

For product-mix auctions,

- The utility functions are tropical polynomials in per-unit prices: $p \mapsto \bigoplus_{b \in A} u(b) \odot x^b$
- For utility functions, tropical hypersurface = LIP (locus of indifference prices).
Aggregation in tropical language

Recall:

▶ The utility functions are tropical polynomials in per-unit prices: \( p \mapsto \bigoplus_{b \in A} u(b) \odot x \odot b \)

▶ For utility functions, tropical hypersurface \( = \) LIP (locus of indifference prices).

Aggregation:

▶ By definition, the aggregate utility is the tropical product, as tropical polynomials, of individual utilities.

▶ Aggregate LIP = union of individual LIPs.
  (Recall: \( \mathcal{I}(F \odot G) = \mathcal{I}(F) \cup \mathcal{I}(G) \).

▶ edge directions in the aggregate demand complex = union of edge directions in individual demand complexes.
Tropical hypersurfaces (LIPs) and demand complexes

- Tropical hypersurfaces (LIPs)
- Demand complexes
Unimodularity Theorem

Let $\mathcal{D}$ be a set of primitive integer vectors in $\mathbb{Z}^n$. A valuation $u$ is of demand type $\mathcal{D}$ if all the primitive edge directions of the demand complex $D_p(u)$ are in $\mathcal{D}$.

A set of vectors $\mathcal{D} \subset \mathbb{Z}^n$ is unimodular if every linearly independent subset of $n$ vectors in $\mathcal{D}$ spans $\mathbb{Z}^n$ over $\mathbb{Z}$.

Theorem (DKM 1998–2003, BK 2014)

The set $\mathcal{D}$ is unimodular if and only if there exists a competitive equilibrium for every collection of concave valuations of demand type $\mathcal{D}$ and for every supply bundle in the domain.

- Elizabeth Baldwin and Paul Klemperer, *Understanding preferences: “demand types”, and the existence of equilibrium with indivisibilities*.

Examples of unimodular demand types

Rent division  There is one item of each type. Supply = 
\((1, 1, \ldots, 1) \in \mathbb{Z}^n\). Each of the \(n\) buyers buys at
most one item, so \(A^j = \{0, e_1, \ldots, e_n\}\) for each
agent \(j\), and
\(D = \{e_1, \ldots, e_n\} \cup \{e_i - e_j : 1 \leq i, j \leq n\}\), which is
(a projection of) the type \(A\) unimodular system.
Rental harmony can always be achieved!

Stable matching with transferable utilities  \(n\) women and \(m\)
men. An “agent” is possible match, on a domain
\(\{0, (e_i, e_j)\} \subset \mathbb{Z}^n \times \mathbb{Z}^m\). The demand set is
\(D = \{(e_i, e_j) : 1 \leq i \leq n, 1 \leq j \leq m\}\), which is
unimodular. Stable matching exists!
The proofs

The necessity of unimodularity is easy: If \( \mathcal{D} \) is not unimodular, there exist linearly independent \( b_1, \ldots, b_k \in \mathcal{D} \) such that

\[
\{0, b_1\} + \cdots + \{0, b_k\} \neq \mathbb{Z}^n \cap \left[\operatorname{conv}\{0, b_1\} + \cdots + \operatorname{conv}\{0, b_k\}\right]
\]

- Take \( A^i = \{0, b_i\} \) and \( u^i \) to be any function.
- The aggregate demand complex is a single parallelopipded \( \operatorname{conv}\{0, b_1\} + \cdots + \operatorname{conv}\{0, b_k\} \) and all its faces.
- The aggregate demand set = \( \{0, b_1\} + \cdots + \{0, b_k\} \), consisting of the vertices of the parallelopipded.
- There are missing integer points inside, by the assumption above.
A proof of the sufficiency of unimodularity (inspired by tropical geometry)

- The existence of competitive equilibrium can be checked one cell at a time in the aggregate demand complex.
- We need to show that each demand set $D$ “has no holes”, i.e. it satisfies $D = \text{conv}(D) \cap \mathbb{Z}^n$.

$$D_p(U) \overset{\text{def}}{=} D_p(u^1) + \cdots + D_p(u^m)$$

$u^i$'s concave

$$= \left[ \text{conv}(D_p(u^1)) \cap \mathbb{Z}^n \right] + \cdots + \left[ \text{conv}(D_p(u^m)) \cap \mathbb{Z}^n \right]$$

$$= \left[ \text{conv}(D_p(u^1)) + \cdots + \text{conv}(D_p(u^m)) \right] \cap \mathbb{Z}^n.$$

**Lemma**

Let $\mathcal{D}$ be a unimodular set of vectors and $P$ and $Q$ be lattice polytopes with edge directions in $\mathcal{D}$. Then

$$(P + Q) \cap \mathbb{Z}^n = (P \cap \mathbb{Z}^n) + (Q \cap \mathbb{Z}^n).$$

**Note:** The “usual” unimodularity theorem in integer programming gives the statement when the facet normal directions, instead of edge directions, are unimodular.
Lemma
Suppose $\mathcal{D}$ is unimodular. If $P$ and $Q$ are lattice polytopes with edge directions in $\mathcal{D}$, then

$$(P + Q) \cap \mathbb{Z}^n = (P \cap \mathbb{Z}^n) + (Q \cap \mathbb{Z}^n).$$

Proof Sketch.

- The inclusion "$\supseteq"$ is easy.
- For "$\subseteq$", if $P$ and $Q$ are contained in complementary affine spaces, then using unimodularity we can choose a lattice basis so that $P$ and $Q$ lie in complementary coordinate subspaces. Then $\text{conv } P + \text{conv } Q = \text{conv } P \times \text{conv } Q$, which has no holes.
- We use the following claim to reduce to the case above.
- **Claim**: For any polytopes $P, Q \subset \mathbb{R}^n$, we can subdivide $P + Q$ into polytopes of the form $F + G$ where $F, G$ are faces of $P, Q$ respectively and $\dim(F) + \dim(G) = \dim(F + G)$. In particular, we do not introduce new edge directions.
- **Proof of Claim**: Take a generic vector $w \in \mathbb{R}^n$ and consider $\{(x, w \cdot x) : x \in P\} + Q \subset \mathbb{R}^{n+1}$. The projection of its upper concave hull gives the desired subdivision. (Intuition: translate each tropical hypersurface a little in a random direction, until they meet transversely.)
Checking the existence of competitive equilibrium

We want to find a \( y \in \{0, 1\}^{(\bigcup_j A^j)} \) such that for \( j^{\text{th}} \) agent and \( a \in A^j \),

\[
y^j_a = \begin{cases} 
1 & \text{if bundle } a \text{ is assigned to } j^{\text{th}} \text{ agent} \\
0 & \text{otherwise}
\end{cases}
\]

A competitive equilibrium exists at supply \( a \) if and only if there is an integer optimal solution to the following linear program with decision variable \( y \):

\[
\begin{align*}
\text{maximize} & \quad y \cdot u \\
\text{subject to} & \quad y \geq 0 \quad \text{and} \quad C \ y = \begin{pmatrix} 1 \\ a \end{pmatrix}
\end{align*}
\]

where \( u \) is the vector of valuations and \( C \) is the Cayley sum

\[
C = \begin{pmatrix} 
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \\
A^1 & A^2 & \cdots & A^m
\end{pmatrix}.
\]
Even when competitive equilibrium exists, it may be difficult to find an equilibrium assignment. The SUBSET-SUM problem is a special case of this.

This IP/LP formulation does not directly give the Unimodularity Theorem.

We have a different (and much less intuitive) reformulation that can be used to derive Unimodularity Theorem from the “usual” unimodularity theorem from integer programming.
Profit optimization at competitive equilibrium

Suppose competitive equilibrium exists at supply bundle $a$. Let $y$ be any optimal solution of the previous LP=IP. Then the following linear program finds a price $p \in \mathbb{R}^n$ that maximizes the profit $p \cdot a$ for the seller.

$$\begin{align*}
\text{maximize} & \quad p \cdot a \\
\text{subject to} & \quad u^j(a) - p \cdot a \geq u^j(b) - p \cdot b,
\end{align*}$$

for every $a \in A^j$ such that $y(a, j) = 1$, and every $b \in A^j$, for $j = 1, \ldots, m$.

- The constraints above give a (non-obvious!) description of the set of equilibrium prices for supply $a$.
- A straightforward formulation involves knowing the cell in the demand complex that contains $a$ in its relative interior, but the above description does not.
- If there are multiple integer optima for the previous IP, then they give different LPs, which all have the same solution.
Summary and open problems

Summary

▶ Some interesting problems in economics have natural formulations in tropical language.
▶ This may give new geometric insights.

Open problems

▶ Is the Cayley-sum structure of the integer program useful?
▶ Find weaker conditions for existence of competitive equilibrium for a given supply bundle. Given lattice polytopes $P$ and $Q$ in $\mathbb{R}^n$, find conditions to guarantee that

$$(P + Q) \cap \mathbb{Z}^n = (P \cap \mathbb{Z}^n) + (Q \cap \mathbb{Z}^n).$$

This has interesting applications in toric and polyhedral geometry (related to “normality” of polytopes).
▶ Online product-mix auction. Can the product-mix auction be modified so that it can be done online, that is, where buyers and products can enter and exit the auction? Are there efficient algorithms to update the price?

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