Example 1

$$y_{n+2} + y_{n+1} - 2y_n = \frac{h}{4}[f(x_{n+2}, y_{n+2}) + 8f(x_{n+1}, y_{n+1}) + 3f(x_n, y_n)].$$

Example 2

$$y_{n+2} - y_{n+1} = \frac{h}{3}[3f(x_{n+1}, y_{n+1}) - 2f(x_n, y_n)].$$

Example 3

$$y_{n+3} + \tfrac{1}{4}y_{n+2} - \tfrac{1}{2}y_{n+1} - \tfrac{3}{4}y_n = \frac{h}{8}[19f(x_{n+2}, y_{n+2}) + 5f(x_n, y_n)].$$

Example 4

$$y_{n+2} - y_n = h[f(x_{n+2}, y^*_{n+2}) + f(x_n, y_n)],$$

where

$$y^*_{n+2} - 3y_{n+1} + 2y_n = \frac{h}{2}[f(x_{n+1}, y_{n+1}) - 3f(x_n, y_n)].$$

Example 5

$$y_{n+1} - y_n = \frac{h}{4}(k_1 + 3k_3),$$

where

$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + \tfrac{1}{3}h, y_n + \tfrac{1}{3}hk_1)$$
$$k_3 = f(x_n + \tfrac{2}{3}h, y_n + \tfrac{2}{3}hk_2).$$

Example 6

$$y_{n+1} - y_n = \frac{h}{2}(k_1 + k_2),$$

where

$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + h, y_n + \tfrac{1}{2}hk_1 + \tfrac{1}{2}hk_2).$$

2.4.1. Apply the method

$$y_{n+2}-y_{n+1}=\frac{h}{12}[4f(x_{n+2},y_{n+2})+8f(x_{n+1},y_{n+1})-f(x_n,y_n)]$$

to the scalar initial value problem $y'=x$, $y(0)=0$ to get a one-step difference equation of the form $y_{n+2}-y_{n+1}=\varphi(n,h)$. By trying a particular solution of the form $y_n=An^2+Bn$, find the exact solution of this difference equation satisfying the initial condition $y_1=\frac{1}{2}h^2$ (which coincides with the exact solution of the problem at $x=h$). Hence show that as $h\to 0$, $n\to\infty$, $x=nh$, the sequence $\{y_n\}$ so obtained does converge, but not to the solution of the initial value problem. Why is this?

2.4.2. Use the method of Exercise 2.4.1 to compute numerical solutions of the scalar initial value problem $y'=4xy^{1/2}$, $y(0)=1$ for $0\leqslant x\leqslant 2$, using the steplengths $h=0.1$, 0.05 and 0.025. Compare the results with the exact solution $y(x)=(1+x^2)^2$ and deduce that the numerical solutions are not converging to the exact solution as $h\to 0$.

2.5.1. Find the range of α for which the method

$$y_{n+2}+(\alpha-1)y_{n+1}-\alpha y_n=\frac{h}{4}[(\alpha+3)f(x_{n+2},y_{n+2})+(3\alpha+1)f(x_n,y_n)]$$

is zero-stable. Apply the method, with $\alpha=-1$ to the scalar initial value problem $y'=y$, $y(0)=1$, and solve exactly the resulting difference equation, taking the starting values to be $y_0=y_1=1$. Hence show that the numerical solution diverges as $h\to 0$, $n\to\infty$.

2.5.2. Quade's method is given by

$$y_{n+4}-\frac{8}{19}(y_{n+3}-y_{n+1})-y_n=\frac{6h}{19}(f_{n+4}+4f_{n+3}+4f_{n+1}+f_n),$$

where $f_{n+j}=f(x_{n+j},y_{n+j})$, $j=0,1,\dots,4$. Show that the method is convergent.

2.5.3. A method is given by

$$y_{n+2}=\hat{y}_{n+2}-6\alpha(y_{n+1}-y_n)+\alpha h(f_{n+2}-4f_{n+3/2}+7f_{n+1}+2f_n)$$

$$\hat{y}_{n+2}=2y_{n+1}-y_n+\frac{h}{3}(4f_{n+3/2}-3f_{n+1}-f_n),$$

where $f_{n+j}=f(x_{n+j},y_{n+j})$, $j=0,1,2$, $f_{n+3/2}=f(x_{n+3/2},y_{n+3/2})$ and $y_{n+3/2}$ is given by a formula of the form

$$y_{n+3/2}+\tilde{\alpha}_1y_{n+1}+\tilde{\alpha}_0y_n=h(\tilde{\beta}_1f_{n+1}+\tilde{\beta}_0f_n).$$

Show that the method satisfies the conditions (2.5) of §2.2 and is consistent. Find the range of α for which it is zero-stable.

2.5.4. Demonstrate the effect of zero-instability by using the method

$$y_{n+2}-(1+\alpha)y_{n+1}+\alpha y_n=\frac{h}{2}[(3-\alpha)f(x_{n+1},y_{n+1})-(1+\alpha)f(x_n,y_n)]$$

with (i) $\alpha=0$, (ii) $\alpha=-5$ to compute numerical solutions of the scalar initial value problem $y'=4xy^{1/2}$, $y(0)=1$ for $0\leqslant x\leqslant 2$, using the steplengths $h=0.1$, 0.05, 0.025.

2.5.5*. The family of methods (2.16) is a sub-family of the two-parameter family of methods

$$y_{n+2} - (1+\alpha)y_{n+1} + \alpha y_n = h[(1+\beta)f_{n+2} - (\alpha+\beta+\alpha\beta)f_{n+1} + \alpha\beta f_n], \tag{1}$$

where $f_{n+j} = f(x_{n+j}, y_{n+j})$, $j = 0, 1, 2$. The family (1) is a useful one for illustrative purposes since it has the property that when it is applied to the scalar initial value problem $y' = y$, $y(0) = 1$ the resulting difference equation can be solved exactly. Show that this solution, with starting values $y_0 = \eta_0\ [= \eta_0(h)]$, $y_1 = \eta_1\ [= \eta_1(h)]$ is

$$y_n = \left[A\alpha^n + \left(\frac{1-\beta h}{1-(1+\beta)h}\right)^n\right]\Big/ C,$$

where

$$A = (-1+\beta h)\eta_0 + [1-(1+\beta)h]\eta_1, \quad B = [1-(1+\beta)h](\alpha\eta_0 - \eta_1), \quad C = \alpha - 1 - (\alpha - \beta + \alpha\beta)h.$$

Show further that

$$\left(\frac{1-\beta h}{1-(1+\beta)h}\right)^n = \begin{Bmatrix} \exp(x_n)[1 + (\frac{1}{2}+\beta)x_n h + 0(h^2)] & \text{if } \beta \neq -\frac{1}{2} \\ \exp(x_n)[1 + \frac{1}{12}x_n h^2 + 0(h^3)] & \text{if } \beta = -\frac{1}{2} \end{Bmatrix}.$$

(*Hint*: Consider the expansion of the logarithm of the left side.) We assume that the starting values satisfy

$$\lim_{h\to 0} \eta_i(h) = 1, \qquad i = 0, 1. \tag{2}$$

(i) Demonstrate that when $|\alpha| < 1$ the method converges, for all starting values satisfying (2).

(ii) Demonstrate that when $|\alpha| > 1$ the method diverges for *general* starting values satisfying (2), but that it converges for the specific starting values $\eta_0 = 1$, $\eta_1(h) = (1-\beta h)/[1-(1+\beta)h]$ (which satisfy (2)). Why would we not be able to demonstrate this numerically? Try doing so.

(iii) Demonstrate that when $\alpha = 1$, there exist some starting values satisfying (2) for which the method converges, and some for which it diverges (sometimes in the sense that $\{y_n\}$ converges to the wrong solution, and sometimes in the sense that $y_n \to \infty$ as $h \to 0$, $n \to \infty$).