RESEARCH STATEMENT

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Overview

Broadly, my research interests are in combinatorial mathematics. My research to date has focused on the combinatorics of partially ordered sets, although I am also interested in extremal problems in other areas of combinatorics. I have been fortunate during my graduate studies to work on problems posed by others as well as questions developed within the W.T. Trotter research group at Georgia Tech. I look forward to continuing my collaborative relationships established at Georgia Tech as well as working with other mathematicians in the coming years. I envision that my future research will continue to blend answering questions posed by others with working on problems I pose. This overview gives a summary of my research interests for those who are not specialists in combinatorics. The subsequent sections provide more detailed information about my research and problems on which I hope to work in the near future.

One of the principal areas on which I have worked is linear discrepancy for partially ordered sets. The problem of linear discrepancy arises naturally when considering how to transform a partial order into a total order that respects the partial order. In applied settings, it is important that incomparable elements of the ground set not be placed too far apart in the total order. If incomparable elements are placed far apart, an implicit comparison is created between them. This comparison unfairly biases against the element that is placed lower, suggesting that it is smaller in the original partial order.

The first linear discrepancy problem on which I worked was completing the characterization of the posets with linear discrepancy 2. We were able to provide a complete list of the forbidden subposets for a poset that has linear discrepancy 2. This list consists of an infinite family of posets along with a small set of isolated examples. Another area of linear discrepancy on which I have worked is providing upper bounds on a poset's linear discrepancy in terms of the maximum number of elements of the ground set to which any element is incomparable. This work attempts to address one of the original questions asked about linear discrepancy. My research has provided tight bounds for two classes of posets—disconnected posets and interval orders.

Another of my research interests is online algorithms for graphs and posets. The conventional framework for algorithmic questions assumes that complete information about the structure being examined is known when the algorithm begins its work. An online algorithm is one that receives information about the elements of the structure one at a time and must make an irrevocable decision about how to treat that element when it arrives. For example, when considering graph coloring online, vertices are revealed one-by-one along with information about which of the previously-revealed vertices are adjacent to the new vertex. The algorithm must then decide how to color the vertex before the next vertex is presented. Online algorithms have been a recurring area of inquiry in our research group, with my results on online algorithms coming in the area of linear discrepancy. We have addressed online algorithms for linear discrepancy in general as well as for the special case of semiorders.

Recently, commutative algebraists working on a longstanding conjecture of R.P. Stanley regarding what is known as Stanley depth have established a connection between their work and posets. Our group has worked to leverage our strength in the combinatorics of partially ordered sets to advance this area of research. We have used special partitions of posets associated to squarefree Veronese ideals to determine the Stanley depth of such ideals. I hope that the research we have completed in this area will lead to continuing this line of research by working with algebraists to combine our individual strengths and develop new results.

1. LINEAR DISCREPANCY FOR PARTIALLY ORDERED SETS

Tanenbaum, Trenk, and Fishburn introduced the concept of linear discrepancy for partially ordered sets in a 2001 paper [18], proposing it as a way to measure the extent to which linear extensions are forced to bias against at least one element. For a poset $\mathbf{P}$, its *linear discrepancy*, denoted $\mathrm{ld}(\mathbf{P})$, is the minimum over all linear extensions of $\mathbf{P}$ of the maximum distance in the linear extension between incomparable points. If $\mathbf{P}$ is a chain, then $\mathrm{ld}(\mathbf{P})$ is defined to be 0. Fishburn, Tanenbaum, and Trenk showed in [5] that determining the linear discrepancy of a poset is **NP**-complete since its linear discrepancy is equal to the bandwidth of its cocomparability graph, which is **NP**-complete to determine by a result of Kloks, Kratsch, and Müller [14].

In [18], the authors showed that if $\mathbf{P}$ is a semiorder, then $\mathrm{ld}(\mathbf{P}) = \mathrm{width}(\mathbf{P}) - 1$. Using this result they were able to characterize the posets $\mathbf{Q}$ for which $\mathrm{ld}(\mathbf{Q}) = 1$ as the semiorders of width 2, thus providing an elegant forbidden subposet characterization of these posets. They then asked if a similar characterization of the posets with linear discrepancy equal to 2 existed. In [15], Rautenbach conjectured that there was in fact a finite list of forbidden subposets that characterized the posets of linear discrepancy 2. The concept of *irreducibility* with respect to linear discrepancy, introduced by Chae, Cheong, and Kim in [4], has proved to be a useful tool for this characterization problem. A poset $\mathbf{P} = (X, \leq_P)$ is k-discrepancy-irreducible if $\mathrm{ld}(\mathbf{P}) = k$ and $\mathrm{ld}(\mathbf{P} - \{x\}) < k$ for all $x \in X$. Howard, Chae, Cheong, and Kim showed in [7] that there are infinitely many 3-discrepancy-irreducible posets of width 2. Since the only 3-discrepancy-irreducible poset of width 4 is the 4-element antichain, it remained only to determine the 3-discrepancy-irreducible posets of width 3. Working with fellow Georgia Tech graduate students D.M. Howard and S.J. Young, I showed in [8] that with a small list of exceptions, all of the 3-discrepancy-irreducible posets of width 3 can be derived from the infinite family for width 2 by removing cover relations. Coupled with a lemma establishing that posets with linear discrepancy greater than 2 must contain a 3-discrepancy-irreducible poset, we were then able to provide a full characterization of the posets with linear discrepancy equal to 2.

My collaboration with S.J. Young on linear discrepancy extended to a second paper [11]. One of the unsatisfying aspects of [8] was the lack of a result that if $\mathrm{ld}(\mathbf{P}) = k$, then $\mathbf{P}$ contains a poset of linear discrepancy j for all $j < k$. Such a result would be expected, since for most monotonic combinatorial parameters, removing a single point can cause only a small change in the parameter's value. However, for linear discrepancy it is possible to remove a point and create a poset with much smaller linear discrepancy. For example, $\mathrm{ld}(\mathbf{1} + \mathbf{n}) = \lceil n/2 \rceil$, but removing the isolated point results in a poset with linear discrepancy 0. Fortunately, a weaker, but still useful, result is true. In [11], we showed that for every poset $\mathbf{P}$, there exists a point which can be deleted without decreasing the linear discrepancy by more than one. We also proved two results about the relationship between critical pairs and linear discrepancy. We showed that the maximum distance in a linear extension between incomparable pairs is achieved only at critical pairs and furthermore that there is a linear extension of $\mathbf{P}$ that is optimal with respect to linear discrepancy and reverses no critical pairs. This is an interesting contrast to dimension, where the concept of critical pairs originated, except with the opposite goal: reversing critical pairs. When determining the linear discrepancy of specific classes of posets, other authors have noted that only certain incomparable pairs need be considered, but we were the first to recognize that these pairs are critical pairs and that the idea generalizes to arbitrary posets.

S.J. Young and I employed these results to address another question of Tanenbaum et al. from [18]. After determining that $\mathrm{ld}(\mathbf{t}+\mathbf{t}) = \lfloor (3t-1)/2 \rfloor$, they asked if every poset in which each point is incomparable to at most t others has linear discrepancy at most $\lfloor (3t-1)/2 \rfloor$. Since t is also the maximum degree of the poset's cocomparability graph, bounds of this form have become known as *degree bounds*, with t denoted by $\Delta(\mathbf{P})$. Rautenbach was the first to improve the initial upper bound of $2\Delta(\mathbf{P})-1$ given in [18]. In [15], he showed that $\mathrm{ld}(\mathbf{P}) \leq 2\Delta(\mathbf{P})-2$. For $\Delta(\mathbf{P}) = 2$ and 3, this also resolves the question of Tanenbaum et al. in the affirmative. In [11], we showed that the answer is "yes" for two additional classes of posets: interval orders and disconnected posets. In fact, for interval orders our result is stronger. We showed that if $\mathbf{P}$ is an interval order, then $\mathrm{ld}(\mathbf{P}) \leq \Delta(\mathbf{P})$ with equality if and only if $\mathbf{P}$ contains an antichain of size $\Delta(\mathbf{P})+1$. It is not difficult to see that the stronger inequality is tight for posets of large width, but large width is not necessary; we showed that there is a family of interval orders of width 2 for which $\mathrm{ld}(\mathbf{P}) = \Delta(\mathbf{P})-1$.

Enough tools have now been assembled to develop a deeper understanding of linear discrepancy, and the area of degree bounds appears to be a good place to start. Since the case $\Delta(\mathbf{P}) = 4$ is the first for which Rautenbach's bound of $2\Delta(\mathbf{P})-2$ differs from the conjectured bound, the investigation of whether $\mathrm{ld}(\mathbf{P}) \leq 5$ for this class of posets may provide greater insight. Another approach would be to improve the upper bound to $(2-\varepsilon)\Delta(\mathbf{P})$ for some $\varepsilon > 0$; perhaps probabilistic techniques could lead to such a bound. The role of critical pairs in linear discrepancy is also interesting. Our proof of the characterization result in [8] also makes use of critical pairs in the poset. A deeper understanding of the relationship between critical pairs and linear discrepancy will be important in addressing another conjecture stated in [18], where it is attributed to Brightwell and Trotter, and also appearing as [19]. This conjecture involves the relationship between linear discrepancy and order dimension. Combining well-known bounds, we have $\dim(\mathbf{P}) \leq \mathrm{ld}(\mathbf{P})+1$. In [19], an example of a 4-dimensional poset for which this inequality is tight is given, but it is conjectured that $\dim(\mathbf{P}) \leq \mathrm{ld}(\mathbf{P})$ for posets with $\dim(\mathbf{P}) \geq 5$. Tanenbaum et al. ask (if the conjecture holds) if posets for which $\mathrm{ld}(\mathbf{P}) = \dim(\mathbf{P}) = n \geq 5$ must contain an induced copy of the standard example $\mathbf{S}_n$. A solution to this challenging conjecture will likely require the use of multiple results from dimension theory and new results for linear discrepancy, integrating this new area of study into the more established field of dimension theory.

2. ONLINE ALGORITHMS FOR GRAPHS AND POSETS

It is convenient to think of an online combinatorial problem as a game between two players called Builder and Assigner. For online problems involving posets, Builder presents the points of the poset one at a time and at each step informs Assigner of the relations between the previously-presented points and the new point. This information must be consistent with that previously presented, e.g., transitivity cannot require that two previously-incomparable points become comparable when a new point is presented. Assigner then makes an irrevocable decision such as where to place the new point in a linear extension or into which chain of a chain partition it should be placed.

In [10], N. Streib, W.T. Trotter, and I considered the problem of online linear discrepancy. Specifically, as Builder presents the points of the poset, Assigner maintains a linear extension of that poset, trying to keep the maximum distance in the linear extension between incomparable pairs as close to the linear discrepancy of the poset as possible. Kloks, Kratsch, and Müller showed in [14] that no linear extension has linear discrepancy worse than three times the poset's linear discrepancy, and so any algorithm Assigner uses will be at worst 3-competitive. This stands in contrast to the online graph bandwidth problem, which Board considered in [2]. He showed that for arbitrary graphs, Builder can force Assigner to construct a permutation of the vertex set with bandwidth far from the optimal value. In [10], we showed that there is a family of interval orders for which, regardless of the algorithm used by Assigner, Builder can force Assigner to construct a linear extension that has linear discrepancy three times the optimal value, provided that Builder is not required to provide an interval representation. For the class of semiorders, however, we showed that there is a 2-competitive online linear discrepancy algorithm. We also show that regardless of Assigner's algorithm, Builder can present a semiorder and force the construction of a linear extension with linear discrepancy twice the optimal value.

There are several other interesting problems involving online algorithms for graphs and posets. The problems of online dimension and online chain partitioning have been studied for many years, but, for example, online chain partitioning is not well understood in general. It is known that Builder can force Assigner to use $\binom{w+1}{2}$ chains for a poset of width w, but the best upper bound on the number of chains that Assigner needs is $(5^w - 1)/4$, which is a result of Kierstead [13] from 1981. Chain partitioning interval orders is equivalent to coloring interval graphs, and determining the competitive ratio of the greedy online algorithm First Fit has long been a question of interest. At present, it is known to be between 4.99 and 8. In [3], Bosek, Krawczyk, and Szczypka have recently investigated the behavior of First Fit on posets which do not contain two incomparable chains of length k and have shown that First Fit needs at most $(4k-2)(w-1)w+w$ chains on such posets of width w. For $k = 2$, the class of posets they considered is the class of interval orders, so their quadratic bound is not not as good as the best known bound. It would be very interesting to see if their bound can be improved to be linear in w for all values of k.

3. Interval Partitions and Stanley Depth

In [17], Stanley introduced the idea of what is now called the Stanley depth of a $\mathbb{Z}^n$-graded module over a commutative ring and conjectured that a module's Stanley depth is always at least its depth. This conjecture remains open in most cases. Herzog, Vladoiu, and Zheng considered the Stanley depth of monomial ideals of $S = K[x_1, \ldots, x_n]$ with K a field in [6]. They showed that if $J \subseteq I$ are monomial ideals of S, then the Stanley depth of I/J can be computed by partitioning a finite subposet of $\mathbb{N}^n$ determined by the generators of I and J into intervals. (An *interval* $[x, y]$ in a poset $\mathbf{P} = (X, \leq_P)$ is the set $\{z \in X \mid x \leq z \leq y\}$.) In the case where the ideals are squarefree, the poset in question can be viewed as a subposet of the Boolean algebra of subsets of $[n] := \{1, 2, \ldots, n\}$. Herzog et al. claimed that for $n \leq 9$, they were able to prove that the Stanley depth of the maximal ideal $(x_1, \ldots, x_n)$ is $\lceil n/2 \rceil$ but could show that this is true for larger values of n. From the combinatorial perspective, the question of Herzog et al. has an appealing restatement: can the poset consisting of the nonempty subsets of $[n]$ be partitioned into a collection $\mathcal{P}$ of intervals so that for all $[X, Y] \in \mathcal{P}$, $|Y| \geq \lceil n/2 \rceil$? In [1], Cs. Biró, D.M. Howard, W.T. Trotter, S.J. Young, and I showed that the answer to this question is "yes," and we were able to give two different proofs yielding nonisomorphic partitions. The result we proved is stronger than needed for the algebraic result and provides a structural view of the poset of nonempty subsets of $[2k + 1]$. (The result for $n = 2k + 2$ follows by a parity argument.) Our first proof is recursive while the second provides a direct construction of the interval which contains any given set.

Shen subsequently showed in [16] that the algebraic version of our result can be extended to show that the Stanley depth of a complete intersection monomial ideal minimally-generated by m monomials is $n - \lfloor m/2 \rfloor$. Shen also proved lower bounds for the Stanley depth of 3- and 4-generated squarefree monomial ideals that are not complete intersection monomial ideals. He then asked if every m-generated squarefree monomial ideal in $K[x_1, \ldots, x_n]$ has Stanley depth at least $n - \lfloor m/2 \rfloor$. S.J. Young and I answered this question in the affirmative in [12], where we offered not only an answer to Shen's question but also a significantly shorter proof that encompassed his results for 3- and 4-generated squarefree monomial ideals. We were able to do this by focusing on the combinatorial view, while Shen had passed repeatedly between the language of posets and that of ideals.

Since the poset we considered in [1] consists of all subsets of $[n]$ of size at least 1, we next considered the problem of partitioning the subposet of the Boolean algebra of subsets of $[n]$ consisting of all subsets of size at least $d \geq 1$ into intervals. This poset corresponds to the squarefree Veronese ideal of degree d in $K[x_1, \ldots, x_n]$, which is also the Stanley-Reisner ideal of the $(d-1)$-skeleton of the n-simplex and is denoted $I_{n,d}$. Y.-H. Shen, N. Streib, S.J. Young, and I addressed this question in [9]. For $d > 1$, we also want intervals $[X, Y]$ for which $|Y|$ is bounded below, but obviously $\lceil n/2 \rceil$ is not attainable. For $d \leq \lfloor n/2 \rfloor$, a counting argument suggests a conjecture that sdepth $I_{n,d} = \lfloor \binom{n}{d+1} / \binom{n}{d} \rfloor + d$. The structure of the subset lattice allows us to consider only n of the form $cd + (c-1)$ for c, d positive integers. Although somewhat counterintuitive, we have found this problem easier to approach from the perspective of showing that for a given c, the conjecture formula is true for all values of d. For $c = 2$, this is equivalent to the existence of a complete matching between the middle two levels of the lattice of subsets of $[2d+1]$. In [9], we showed that the conjectured formula for sdepth $I_{n,d}$ holds for $n = cd + (c-1)$ when $c \leq 4$. This gives a formula for sdepth $I_{n,d}$ that holds for n, d with $1 \leq d \leq n < 5d + 4$. Our proof uses a construction that generalizes the constructive proof in [1] and involves lemmas about this construction that suggest the conjecture to be true in general. I am not optimistic that an inductive approach such as we used in [1] will be successful even for $d = 2$, as the omission of more than just the empty set creates much larger difficulties to overcome.

Beyond continuing to work on the Stanley depth of squarefree Veronese ideals, this line of research presents many other opportunities. At this point, most of the focus has been on squarefree ideals, allowing us to restrict consideration to subposets of the Boolean algebra of subsets of $[n]$. While the more general context will be less well-understood at the outset, research on interval partitioning problems and their application to Stanley depth may provide further results that are of combinatorial interest in their own right. One challenge we have faced so far is that the combinatorists and algebraists interested in this problem have not been able to interact as much as is needed to make significant progress. Hopefully if my fellow combinatorists and I learn more about the relevant algebra and the interested algebraists learn more about the relevant combinatorics, fruitful collaborations can develop results of interest to both communities.

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