Typical Ranks of Semisimple Graphs

Daniel Irving Bernstein∗  Grigoriy Blekherman†  Kisun Lee‡
∗Massachusetts Institute of Technology  †Georgia Institute of Technology

Introduction

Let $M$ be an $n \times n$ partially-filled symmetric matrix in $S_n(C)$ or $S_n(R)$.

→ complete $M$ in a way that the completion $\overline{M}$ has the lowest possible rank (full-rank typicality).

**Ex.**

\[
M = \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}, \quad x = 0 \implies \text{rank}(\overline{M}) = 0
\]

$M$ is a partial matrix in $S_n(C)$.

→ complete $M$ in a way that

(1) stable under a perturbation
(2) lowest possible rank (generic completion rank of $M$).

**Ex.**

\[
M = \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \implies \text{rank}(\overline{M}) \geq 1
\]

\[
M' = \begin{bmatrix} 1 \pm \varepsilon_1 & x \\ x & 1 \pm \varepsilon_2 \end{bmatrix} \implies \text{rank}(\overline{M'}) \geq 1 \implies \text{gcr}(M) = 1.
\]

Playground!

**Def.** A semisimple graph $G = (V, E)$ is called full-rank typical if $|V|$ is a typical rank of $G$.

**Thm** (Characterization of full-rank typical graphs).
A semisimple graph $G$ is full-rank typical if and only if the complement $G^C$ of $G$ is bipartite.

**A.**

$G^C = \begin{cases} \text{full-rank typical} & \text{if } G \text{ is bipartite} \\
\text{not full-rank typical} & \text{if } G \text{ is not bipartite} \end{cases}$

**Q.** Are $G$ and $H$ full-rank typical?

**B.**

$G^C = \begin{cases} \text{full-rank typical} & \text{if } G \text{ is bipartite} \\
\text{not full-rank typical} & \text{if } G \text{ is not bipartite} \end{cases}$

**Q.** What are the maximal typical rank of $G$ (star tree) and $H$ (tree)?

**Prop.** Let $G$ be an all-looped semisimple graph. Then the maximal typical rank of $G$ is 3 if and only if $G$ is either

(1) a triangle or
(2) a disjoint union of a star tree and a (possibly empty) set of isolated vertices.

**Thm** (Typical ranks of all-looped trees). All-looped trees have typical rank at most 4.

**A.**

- (maximal typical rank of $G$) = 3
- (maximal typical rank of $H$) = 4

For $M \in S_n(R)$, we call such a rank as a typical rank.

Encode $M$ by a semisimple graph $G$ such that $M_{ij} = M_{ji}$ is specified if $(i, j) \in E(G)$.

**Ex.**

\[
M = \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \implies \text{rank}(\overline{M}) \geq 1 \implies (\text{minimal typical rank}) = 1
\]

\[
M = \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \implies \text{rank}(\overline{M}) \geq 2 \implies (\text{maximal typical rank}) = 2
\]

**Prop** (Prop 6.1. Bernstein-Blekherman-Sinn 2018). Let $G = (V, E)$ be a semisimple graph with $|V| = n$. Then,

(1) $\text{gcr}(G)$ exists (completion rank of almost all $G$-partial matrix)
(2) $\text{gcr}(G) \geq k$ if $nk - (\binom{n}{2}) \geq |E|
(3) the smallest typical rank of $G = \text{gcr}(G)$
(4) If $r_1 < r_2$ are typical ranks of $G$, then so is $r$ such that $r_1 \leq r \leq r_2$
(5) (maximal typical rank of $G) \leq 2 \cdot \text{gcr}(G)$

**Remark.** By (3) – (5), in order to know all typical ranks, it is enough to find the generic completion rank and maximal typical rank.

**Prop.** The typical ranks of $K_n^+ \cup K_m^+$ are $\max\{n, m\}, \ldots, n + m$.

**Def.** Given real symmetric matrices $A$ and $B$ of full-rank (possibly different size), we define the eigenvalue sign disagreement of $A$ and $B$ as:

\[
esd(A, B) = \begin{cases} 0 & \text{if } (p_A - p_B)(n_A - n_B) \geq 0 \\
\min\{|p_A - p_B|, |n_A - n_B|\} & \text{otherwise} \end{cases}
\]

where $p, (n)$ is the number of positive (negative) eigenvalues.

**Thm.** Let $M = \begin{bmatrix} A & X \\ X^T & B \end{bmatrix}$ be a $K_n^+ \cup K_m^+$-partial matrix

where $A$ is an $n \times n$-matrix, $B$ is an $m \times m$-matrix which are full-rank and $X$ is an $n \times m$-matrix with unspecified entries. Then, $M$ is minimally completable to rank max$(n, m) + \text{esd}(A, B)$.

**Thm.** Let $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$ be a $K_n^+ \cup K_m^+$-partial matrix

where $A$ is an $n \times n$-matrix, $B$ is an $m \times m$-matrix which are full-rank and $X$ is an $n \times m$-matrix with unspecified entries. Then, $M$ is minimally completable to rank max$(n, m) + \text{esd}(A, B)$.

**Q.** Find the upper bound of the typical rank of $G$.

**Prop.** Let $G$ be an all-looped semisimple graph and $r$ be the size of the maximum independent set of $G$. If $G$ has $n$ vertices, then the maximal typical rank of $G$ is at most $2 + (n - r)$. 

**A.**

- (typical rank) $\leq 2$
- (typical rank) $\leq 3$
- (typical rank) $\leq 4$