Nonnegative Quadratics over Stanley Reisner Varieties

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Background on Nonnegative and Sum-of-Squares Polynomials

Nonnegative Polynomials

Definition

A polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]_d$ is **nonnegative** if for all $x \in \mathbb{R}^n$, $f(x) \geq 0$.

Example

$$f = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)^n - \prod_{i=1}^{n} x_i^2$$

is nonnegative.

Nonnegative Polynomials

Definition

A convex cone is $C \subseteq \mathbb{R}^k$ so that if $x, y \in C$, $x + y \in C$ and if $\lambda \geq 0$, then $\lambda x \in C$.

Nonnegative polynomials form a closed convex cone.

Optimization over slices of the cone of nonnegative polynomials expresses many difficult problems. For this reason, it is considered **hard** to check, given a polynomial, whether or not it is nonnegative.

Algebraic Certificates

One value of working with polynomials is that we can seek out algebraic sufficient conditions for nonnegativity. Such a sufficient condition is be a sum-of-squares.

Definition

A polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]_{2d}$ is **sum-of-squares** (SOS) if there are $g_1, \ldots, g_k \in \mathbb{R}[x_1, \ldots, x_n]_d$ so that

$$f = \sum_{i=1}^{k} g_i^2.$$

Being sum-of-squares is considered **easy** to check, using connections to the field of semidefinite programming.

The AM-GM inequalities can all be proven by using sum-of-squares.

Algebraic Certificates

If $d = 1$, or $n = 2$, or $d = 2$ and $n = 3$, then a polynomial is nonnegative if and only if it is sum-of-squares.

For all other values of n and d, it is known that there exist nonnegative polynomials that are not sum-of-squares.

Gaps between Nonnegative and Sum-Of-Squares

The first nonnegative polynomial that was proven not to be SOS was the Motzkin polynomial

$$x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2.$$

Since then, it has been shown that for large n, almost all nonnegative polynomials are not sums-of-squares [1].

This suggests that sum-of-squares are limited in their ability to approximate nonnegative polynomials for large numbers of variables. **What can we do to better understand nonnegativity?**

Nonnegative Polynomials on Varieties

We will introduce a more general setting. Let X be a real projective variety (it is okay to think of it as a subset of $\mathbb{R}^n$ cut out by homogeneous polynomial equations).

Definition

A polynomial $f \in \mathbb{R}[X]_{2d}$ is **nonnegative** if for all $x \in X$, $f(x) \geq 0$. We denote the convex cone of nonnegative quadratics over X by $\mathcal{P}(X)$.

Definition

A polynomial $f \in \mathbb{R}[X]_{2d}$ is **sum-of-squares** if there are $g_1, \ldots, g_k \in \mathbb{R}[X]_d$ so that

$$f = \sum_{i=1}^{k} g_i^2.$$

We denote the convex cone of nonnegative quadratics over X by $\Sigma(X)$.

Nonnegative Polynomials on Varieties

We now can trade off the complexity of the variety X with the degree d to find tractable examples of this problem. We can usually just consider quadratic polynomials over varieties.

Example

$\mathbf{P}^n$ - Usual theory of nonnegative quadratics in n variables, so $\mathcal{P}(\mathbf{P}^n) = \Sigma(\mathbf{P}^n)$

$\mathbf{P}^n \times \mathbf{P}^m$ - Quadratics over this variety correspond to maps between spaces of symmetric matrices. The nonnegative quadratics correspond to maps preserving the PSD cone, and the sum-of-squares cone correspond to completely positive maps, which are used in quantum information.

$\mathrm{Gr}(4, 2)$ - Quadratics over this variety have connections to Riemannian metrics over spheres, and these have seen applications.[3]

Nonnegative Polynomials on Varieties

There is a complete characterization for varieties where the nonnegative and sum-of-squares quadratics are equal.[2] These are precisely those with Castelnuovo-Mumford regularity 2.

What happens with other kinds of varieties? We will focus on a very simple kind of algebraic variety that turns out to have some surprising geometric properties.

Stanley-Reisner Varieties

Stanley-Reisner Varieties

A Stanley-Reisner variety is the variety cut out by a square-free-monomial ideal. Precisely, for some $\mathcal{S} \subseteq 2^{[n]}$, we have

$$X = \mathcal{V}(\langle \prod_{i \in T} x_i : T \in \mathcal{S} \rangle).$$

We can equivalently think of a Stanley-Reisner variety as being the union of coordinate planes, so that for some $\Delta \subseteq 2^{[n]}$,

$$X = \bigcup_{T \in \Delta} \operatorname{span}\{e_i : i \in T\}.$$

Stanley-Reisner Varieties and Simplicial Complexes

A Stanley-Reisner variety can be encoded in the combinatorial data of which coordinate subspaces are contained in the variety.

Definition

A **simplicial complex** Δ is a family of subsets of $[n]$ with the property that for any $S \in \Delta$, and any $T \subseteq S$, $T \in \Delta$.

We call $S \in \Delta$ a **face** of Δ, and we call elements of $[n]$ the **vertices** of Δ.

Stanley-Reisner Varieties and Simplicial Complexes

The topological realization of a simplicial complex is obtained by gluing together simplices using the simplicial complex as a guide. This topological realization can be visualized by drawing simplices for each set that is contained in the complex.

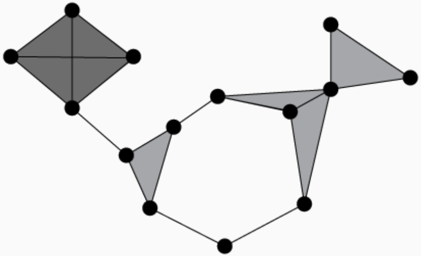


Figure 1: This simplicial complex would be difficult to write down as a family of sets, but this visualization give us a compact representation.

Stanley-Reisner Varieties and Simplicial Complexes

The Stanley-Reisner variety corresponding to a complex Δ is

$$\mathcal{V}(\Delta) = \bigcup_{S \in \Delta} \operatorname{span}(\{e_i : i \in S\}).$$

For $S \in \Delta$, we will call the coordinate subspace $\operatorname{span}(\{e_i : i \in S\}) \subseteq \mathcal{V}(\Delta)$ a face of $\mathcal{V}(\Delta)$.

Nonnegative Quadratics over Stanley-Reisner Varieties

Quadratics over Stanley-Reisner Varieties and PSD Matrix Completion

Quadratics over $\mathbf{P}^n$ can be represented by symmetric matrices that encode their coefficients. Quadratics over a Stanley-Reisner variety can be represented by **partial matrices**, where we project onto the subset of the entries of the matrix that correspond to quadratic monomials not contained in the ideal of X.

Example

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2. \qquad \begin{pmatrix} 1 & -1 & ? \\ -1 & 1 & ? \\ ? & ? & 1 \end{pmatrix}$$

A quadratic form over $X = \mathcal{V}(\langle x_1x_3, x_2x_3 \rangle)$, and a representing partial matrix.

Quadratics over Stanley-Reisner Varieties and PSD Matrix Completion

Sum-of-squares quadratics over X are precisely those that extend to nonnonegative quadratics over $\mathbf{P}^n$, so these correspond to PSD-completable partial matrices.

Nonnegative quadratics over X are only nonnegative on certain subspaces of $\mathbf{P}^n$, so these can be thought of as partial matrices where certain submatrices are PSD.

These two convex cones are useful in sparse semidefinite programming, since it is often easier to check the nonnegativity condition on the small subspaces than the whole of $\mathbf{P}^n$.

Extreme Nonnegative Quadratics over Stanley-Reisner Varieties

Extreme nonnegative quadratics are nonnegative quadratics on Stanley-Reisner varieties that cannot be written as the sum of two linearly independent nonnegative quadratics.

Extreme rays of a convex cone are in some senses the ‘irredudcable’ elements of the cone, and much can be understood about the cone from their structure. We will be investigating how the extreme rays of the nonnegative quadratic cone over Stanley-Reisner varieties relate to the geometric structure of its underlying simplicial complex.

Extreme nonnegative quadratics over Stanley-Reisner Varieties

Over $\mathbf{P}^n$, extreme nonnegative quadratics are exactly squares of linear forms. We say that such a quadratic form has rank 1.

For any variety, the square of a linear form will be extreme. For q to be extreme in $\mathcal{V}(\Delta)$, it suffices for the restriction of q to any faces of $\mathcal{V}(\Delta)$ to be extreme.

Example

Let $\Delta = 2^{[3]} - \{1, 2, 3\}$ be an empty triangle complex. Let $X = \mathcal{V}(\Delta)$.

$$q = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$$

becomes rank 1 when restricted to any 2-dimensional coordinate subspace, but, it is not the square of any linear form.

We call such a quadratic form which is rank 1 when restricted to any face of $\mathcal{V}(\Delta)$ locally rank 1.

Extreme nonnegative quadratics over Stanley-Reisner Varieties

A sum-of-squares quadratic on $\mathbf{P}^n$ is rank k when it is the sum of at most k squares of linear forms.

We say that $q \in \mathcal{P}(\mathcal{V}(\Delta))$ is locally rank k if for any face of $\mathcal{V}(\Delta)$, q restricted to that face of $\mathcal{V}(\Delta)$ is rank k.

Question: What are the possible ranks of extreme nonnegative quadratics on $\mathcal{V}(\Delta)$?

Some examples

Let $\Delta = \binom{[n]}{\leq n-1}$ be the simplicial complex consisting of all sets of size at most $n-1$ in $[n]$. For $n \geq 2$, consider

$$q = \sum_{i=1}^{n} x_i^2 - \frac{2}{n-2} \sum_{i \neq j} x_i x_j$$

q can be seen to have local rank $n-2$, and is also extreme.

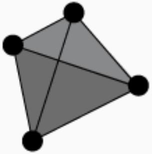


Figure 2: A depiction of $\binom{[4]}{\leq 3}$, which is topologically a 2D sphere.

All extreme rays of $\mathcal{P}(\mathcal{V}(\Delta))$ are either locally rank $n-2$, or the square of a linear form. Interesting that topologically, this is an $n-2$ dimensional sphere. Is there a connection here?

Some examples

Δ is said to be purely 1-dimensional if every maximal face has 2 elements. These are in natural correspondance with graphs.

Theorem

$q \in \mathcal{P}(\mathcal{V}(\Delta))$ is extreme nonnegative if and only if it is locally rank 1.

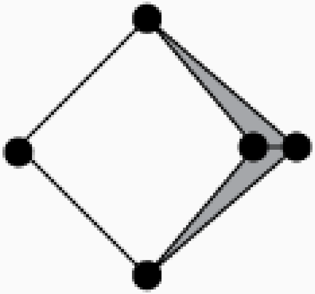
Idea: write

$$q = \sum_{i=1}^{n} a_i x_i^2 + \sum_{i \neq j} a_{ij} x_i x_j$$

If for some facet $\{i, j\}$ of $\mathcal{V}(\Delta)$, q is not rank 1, then we can add and subtract a small amount from a_{ij}, and keep q nonnegative, which contradicts the fact that q is extreme.

Some examples

Δ is given by the following picture:



Δ is not a purely 1-dimensional complex, but it only has locally rank 1 extreme rays. Is this related to the fact that Δ is homotopy equivalent to a purely 1-dimensional complex?

A Categorical Viewpoint

A Categorical Viewpoint

- $\mathcal{V}$ turns a simplicial complex into a real projective variety.
- $\mathcal{P}$ and Σ turn real projective varieties into convex cones.

We are considering the composition of these two transformations:

$$\Delta \mapsto \mathcal{V}(\Delta) \mapsto \mathcal{P}(\mathcal{V}(\Delta)).$$

Idea: Enrich the classes of simplicial complexes and varieties into a category, and these transformations into functors. Then study the nonnegative quadratics using combinatorial maps between simplicial complexes.

Category of Simplicial Complexes

Let Δ and Γ be simplicial complexes. If $\phi : \Delta \to \Gamma$ is a map from the vertices of Δ to vertices of Γ, we say that ϕ is **simplicial** if the image of every face of Δ is a face of Γ.

Simplicial maps make the class of simplicial complexes into a category.

$\mathcal{V}$ as a functor

Let $\phi : \Delta \to \Gamma$ be simplicial. Let $\mathcal{V}(\phi)$ be a linear map from $\mathcal{V}(\Delta)$ to $\mathcal{V}(\Gamma)$ given by extending the following map by linearity:

$$\mathcal{V}(\phi)(e_i) = e_{\phi(i)}.$$

This makes $\mathcal{V}$ into a functor from the category of simplicial complexes to the category of real algebraic varieties and linear maps.

$\mathcal{P}$ as a functor

Let $\phi : X \to Y$ be a linear map between real projective varieties. Then, consider the pullback map ϕ^* on the level of quadratic forms, so that

$$\phi^*(q)(x) = q(\phi(x)).$$

It can easily be seen that thus sends $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ linearly. This makes $\mathcal{P}$ into a contravariant functor from the category of real projective varieties with linear maps to convex cones with linear maps.

Idea: We now can turn any simplicial map from Δ to Γ into a linear map from $\mathcal{P}(\mathcal{V}(\Delta))$ to $\mathcal{P}(\mathcal{V}(\Gamma))$. Can we use this to study nonnegative quadratics?

Constructions of Extreme Rays with High Rank

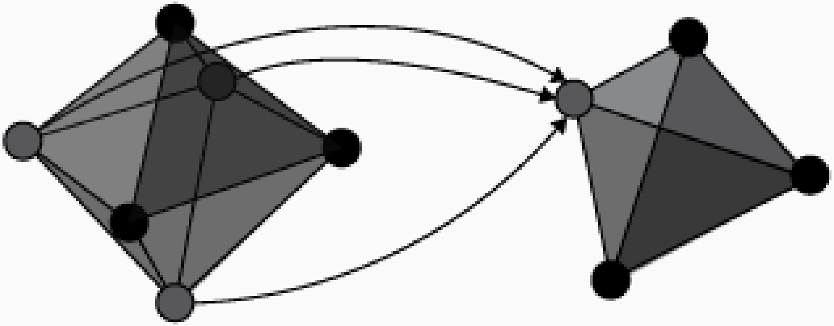
Theorem

Let Δ be a simplicial complex which is an d dimensional combinatorial manifold . Then there is a locally rank d extreme ray of $\mathcal{P}(\mathcal{V}(\Delta))$.

Loosely, the topological realization of Δ should be a manifold, and there should be an atlas of that manifold compatible with the simplicial complex structure.

Proof idea: The hypothesis implies that Δ has a particularly nice simplicial map to $\binom{[d+2]}{\leq d+1}$, which sends all but one face of Δ to a single vertex. The pullback of the locally rank d extreme ray of $\binom{[d+2]}{\leq d+1}$ is locally rank d and extreme in $\mathcal{P}(\mathcal{V}(\Delta))$.

A depiction of a nice map from an octahedral complex to $\binom{[4]}{\leq 3}$.



Chordal Complexes

Definition

The clique complex of a graph G is

$$\Delta = \{S \subseteq V(G) : S \text{ induces a clique of } G\}.$$

Aside: This is right adjoint to the forgetful functor that sends a simplicial complex to the graph consisting of all its 1-dimensional faces.

Definition

A simplicial complex is said to be chordal if it is the clique complex of a graph with no induced cycles of size at least 4.

Chordal Complexes

A theorem of Froberg implies that $\mathcal{X}(\Delta)$ has Castelnuovo-Mumford regularity 2 if and only if it is chordal, so chordal complexes are precisely those where sums-of-squares and nonnegative quadratics are equal. This suggests that the nonnegative quadratics on a chordal complex are significantly better behaved than nonnegative quadratics over nonchordal complexes.

Idea: Take a general simplicial complex Δ and express it as a quotient of a chordal complex. The pullback map will let us study nonnegative quadratics over Δ as sum-of-squares quadratics over a chordal complex.

Chordal Complexes

Definition

A simplcial map $\phi : \Gamma \to \Delta$ is a chordal quotient if Γ is chordal, and every face of Δ is the image of some face of Γ.

Theorem

A simplcial map $\phi : \Gamma \to \Delta$ is a chordal quotient, and then every extreme ray of $\mathcal{P}(\mathcal{V}(\Delta))$ is has local rank at most

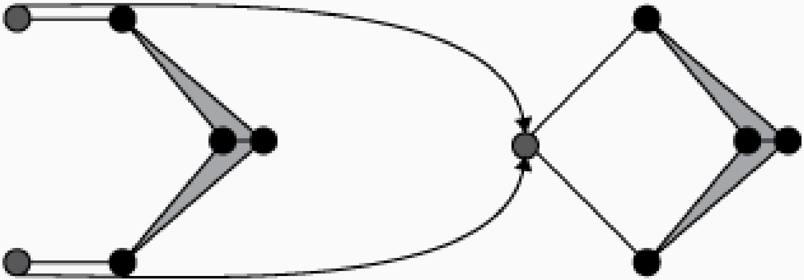
$$|\Gamma| - |\Delta|,$$

where $|\cdot|$ denotes the number of vertices of a simplicial complex.

Idea: If q is extreme in $\mathcal{P}(\mathcal{V}(\Delta))$, then pull q back to $\mathcal{P}(\mathcal{V}(\Gamma))$ and then use the sum-of-squares structure to find a decomposition (this can be done using dimension counting).

Chordal Complexes

The above example is the image of a chordal quotient that uses at most 1 extra vertex, and this implies that every extreme ray of this complex has local rank 1.



A More General Class

In fact, there is a much more general class of simplicial complexes that have only locally rank 1 extreme rays: we show that these can be constructed by starting with a graph, and replacing the edges of that graph by chordal complexes. [4]

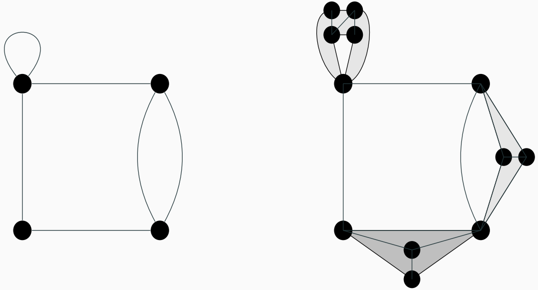


Figure 3: An example of a thickened graph. To the left, is a graph, and to the right is a thickening, where some of the edges have been replaced by other chordal graphs.

Open Questions

- Does every complex with $H^i(\Delta, \mathbb{Z}) \neq 0$ for some i have an extreme ray with local rank at least i? This weakens the assumptions in our theorem on combinatorial manifolds.
- Can we find appropriate generalizations of these ideas that work for other projective varieties?

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