Let $X$ be a finite CW complex. We saw last time that the Spanier-Whitehead dual of $X$ is equivalent to $\Sigma^{-(n-1)}(S^n - X)$ where $X \to S^n$ is any non-surjective embedding of $X$ into a sphere, and such an embedding always exists.

Such an embedding gives an embedding of $X$ in $\mathbb{R}^n$ by removing a point of $S^n$. Adding this point to $X$, we have the formula

$$\mathbb{D}(X_+) \cong \Sigma^{-(n-1)}(\mathbb{R}^n - X).$$

(1)

We may also replace $\mathbb{R}^n$ by the homeomorphic open disk $D^n$.

Note that

$$(\mathbb{R}^n - X) \to \mathbb{R}^n \to \Sigma(\mathbb{R}^n - X)$$

is a cofiber sequence, because $\mathbb{R}^n$ is contractible. Thus we may rewrite (1) as $\mathbb{D}(X_+) \cong \Sigma^{-n}(\mathbb{R}^n/(\mathbb{R}^n - X))$. For any neighborhood $N$ of $X$ in $\mathbb{R}^n$, we thus also have $\mathbb{D}(X_+) \cong \Sigma^{-n}N/(N - X)$.

A sufficiently small neighborhood $N$ deformation retracts back to $X$. Furthermore, if $X$ is a manifold, $N$ can be chosen so as to identify with the disk bundle $D(N_X \mathbb{R}^n)$ of the normal bundle $N_X \mathbb{R}^n \to X$ with $\partial N$ identified with the sphere bundle $S(N_X \mathbb{R}^n)$. Such an $N$ is a tubular neighborhood. We then have $N/(N - X) \cong N/\partial N \cong D(N_X \mathbb{R}^n)/S(N_X \mathbb{R}^n) \cong \text{Th}(N_X \mathbb{R}^n)$. Combining with the previous we obtain

$$\mathbb{D}(X_+) \cong \Sigma^{-n} \text{Th}(N_X \mathbb{R}^n)$$

(2)

Recall that if we add a trivial bundle $\mathbb{R} \to X$ to a vector bundle $V \to X$, the resulting thom space $\text{Th}(V \oplus \mathbb{R}) \cong \Sigma \text{Th}(V)$. This leads to the definition $\text{Th}(V \oplus \mathbb{R}^n) \cong \Sigma^n \text{Th}(V)$ for $n$ positive or negative. Note that the Whitney sum of the tangent bundle and the normal bundle of $X \to \mathbb{R}^n$ is the trivial bundle of rank $n$, i.e. $N_X \mathbb{R}^n \oplus T_X \cong \mathbb{R}^n$. Rearranging terms, we have:

**Theorem 1.1.** If $X$ is a compact manifold without boundary, $\mathbb{D}(X_+) \cong \text{Th}(-T_X)$. 

This implies Poincaré duality [H, Theorem 3.30], but to see this, we need to take as given the Thom isomorphism theorem, which says that if \( V \to X \) is an orientable vector bundle of rank \( n \), then there is a natural isomorphism \( H^{++n}(\text{Th}(V), \mathbb{Z}) \cong H^*(X, \mathbb{Z}) \). One way to think about this is that orientable is a condition which says that \( V \) behaves like a trivial bundle after taking cohomology, so \( \tilde{H}^*(\text{Th}(V), \mathbb{Z}) \cong \tilde{H}^*(\Sigma^n X, \mathbb{Z}) \). Furthermore, for any abelian group \( A \), there is a notion of \( A \)-orientable, and a natural isomorphism \( \tilde{H}^{++n}(\text{Th}(V), A) \cong H^*(X, A) \) when \( V \) is \( A \)-oriented. Furthermore, for any spectrum \( E \), there is a notion of \( E \)-orientable, and a natural isomorphism \( E^{++n}(\text{Th}(V)) \cong E^*(X) \) when \( V \) is \( E \)-oriented. We'll discuss this later.

**Corollary 1.2. (Poincaré duality)** Suppose that \( X \) is a compact \( n \)-manifold without boundary and \( E \) is a spectrum such that the tangent space \( T_X \) of \( X \) is \( E \)-orientable. Then

\[
E_*(X) \cong E^{n-*}(X_+).
\]

In particular, if \( X \) is orientable, then \( H_*(X, \mathbb{Z}) \cong H^{n-*}(X, \mathbb{Z}) \).

**Proof.** Since \( D(X) \cong \text{Th}(-T_X) \), we have that \( [D(X_+), E_+ \cong [\text{Th}(-T_X), E_+] \). The right hand side is \( E^{-*}\text{Th}(-T_X) \). By the Thom isomorphism theorem,

\[
E^{-*}\text{Th}(-T_X) \cong E^{-*}r(-T_X)(X_+) \cong E^{n-*}X_+.
\]

The left hand side is \( [D(X_+), E_+ \cong [S^0, (X_+) \wedge E_+ \cong E_*(X+) \). 

Via the formula \( \text{Th}(R^n) \cong S^n \wedge X_+, X_+ \) can be interpreted as the Thom space of the trivial 0-dimensional bundle. We can generalize Theorem 1.1 with a formula, due to Atiyah [At], for the dual of any Thom space over \( X \). For a compact \( n \)-manifold with boundary, view the tangent bundle as a rank \( n \) vector bundle together with a distinguished sub-bundle of rank \( n-1 \) when restricted to the boundary.

**Theorem 1.3. (Atiyah duality)** If \( X \) is a compact manifold with boundary \( \partial X \) then

\[
D(X/\partial X) \cong \text{Th}(-T_X).
\]

If \( X \) is a compact manifold without boundary and \( V \) is a smooth vector bundle, then

\[
D(\text{Th}(V)) \cong \text{Th}(-T_X - V).
\]

**Proof.** Embed \( X \) into the Euclidean \( n \)-disk \( D^n \), so that \( \partial X \) is embedded into \( \partial D^n = S^{n-1} \). Assume these embeddings are cellular and that \( X \) is transverse to \( S^{n-1} \). Choose a tubular neighborhood \( N \) of \( X \) such that \( N \cap S^{n-1} \) is a tubular neighborhood of \( \partial X \).
$X/\partial X$ is homotopy equivalent to $Y = X \cup \mathcal{C} \cup \partial X \subset D^n \cup \mathcal{C} S^{n-1} \cong S^n$, where as before $\mathcal{C}$ denotes the (unreduced) cone. Applying Alexander duality to $Y \subset S^n$, we have

$$D(X/\partial X) \cong \Sigma^{-(n-1)}(S^n - Y).$$

Note that $N = N \cup \mathcal{C}(N \cap S^{n-1})$ is a neighborhood of $Y$, which deformation retracts onto $Y$. Thus $S^n - Y \cong (S^n - N)$. Since the cone point is not in $(S^n - N)$, we have that $(S^n - N) \cong D^n - N$.

Thus

$$D(X/\partial X) \cong \Sigma^{-(n-1)}(S^n - Y)$$

$$\cong \Sigma^{-n}\Sigma'(S^n - Y)$$

$$\cong \Sigma^{-n}\Sigma'(S^n - N)$$

$$\cong \Sigma^{-n}\Sigma'(D^n - N)$$

$$\cong \Sigma^{-n}\Theta(N X D^n)$$

$$\cong \Theta(-T_X)$$

The second assertion follows from the first. Let $D(V)$ and $S(V)$ denote the disk and sphere bundles of $V$ respectively. Then $D(V)$ is a compact manifold with boundary $S(V)$, so $D(\Theta(V)) \cong \Theta(-T_D(V))$. The kernel of $T_V \to T_X$ is the fiber-wise tangent bundle, which is $V$. Let $v : D(V) \to X$ denote the restriction of $V \to X$ to $D(V)$. Then there is a short exact sequence $v^*V \to T_D(V) \to v^*T_X$.

It turns out that up to homotopy the Thom space of a vector bundle only depends on the bundle’s class in $K$-theory. Thus $\Theta(-T_D(V)) \cong \Theta(-v^*(V \oplus T_X))$. Since $v$ is a homotopy equivalence, $\Theta(-v^*(V \oplus T_X)) \cong \Theta(-T_V \oplus T_X)$, as claimed.

**Example 1.4. Truncated projective space** Recall that $X = \mathbb{R}P^n = S^n/(v \sim -v)$ and that the tautological line bundle $L \to \mathbb{R}P^n$ is the vector bundle whose fiber over $v \in S^n$ is $\ell = \mathbb{R}v$.

Put the standard inner product on $\mathbb{R}^{n+1}$, and let $\ell^\perp$ donate the $n$-dimensional subspace of $\mathbb{R}^{n+1}$ of those vectors perpendicular to $\ell$. Then $T_{[v]}\mathbb{R}P^n \cong \text{Hom}(\ell, \ell^\perp)$. Since $\ell \oplus \ell^\perp \cong \mathbb{R}^{n+1}$, we have

$$0 \to \mathbb{R} = \text{Hom}(\ell, \ell) \to \text{Hom}(\ell, \mathbb{R}^n) \to \text{Hom}(\ell, \ell^\perp) \to 0.$$

This globalizes to an exact sequence

$$0 \to \mathbb{R} \to (L^*)^{n+1} \to T_{\mathbb{R}P^n} \to 0$$

of vector bundles on $\mathbb{R}P^n$. 

3
It turns out that up to homotopy the Thom space of $V$ only depends on the class of $V$ in $K$-theory. Then it follows that

$$\text{DRP}^n \cong \text{Th}(-\text{RP}^n) \cong \text{Th}(\mathbb{R} - (L^*)^{n+1}).$$

We can identify $\text{Th}(L^*)^k$ with a truncated projective space for any $k$ as follows. Any linear function from $\ell \in \mathbb{R}^{n+1}$ to $\mathbb{R}^k$ determines a graph in $\mathbb{R}^k \times \mathbb{R}^{n+1}$, which in turn determines an element of $(\mathbb{R}^k \times \mathbb{R}^{n+1} - \{0\})/\mathbb{R}^* \cong \mathbb{RP}^n + k$. The only elements of $\mathbb{RP}^n + k$ not equal to such a graph, are lines in $\mathbb{R}^k$. Thus we have a vector bundle

$$V = \mathbb{RP}^{n+k} - \mathbb{RP}^{k-1} \rightarrow \mathbb{RP}^n$$

whose fiber over $\ell \in \mathbb{RP}^n$ is $\text{Hom}(\ell, \mathbb{R}^k)$. Thus $V = (L^*)^k$.

$\text{Th}((L^*)^k)$ is the one point compactification of $(L^*)^k$ because $X$ is compact. Thus

$$\text{Th}((L^*)^k) \cong \mathbb{RP}^{n+k}/\mathbb{RP}^{k-1}.$$ 

There is notation for $\mathbb{RP}^{n+k}/\mathbb{RP}^{k-1}$. Define $\mathbb{RP}^{n+k} = \mathbb{RP}^{n+k}/\mathbb{RP}^{k-1}$.

In total, we obtain

$$\text{D}(\mathbb{RP}_+^n) \cong \Sigma \mathbb{RP}^{-1}_{-(n+1)}.$$

References


[M] Haynes Miller, *Vector Fields on spheres, etc. (course notes).*