Lecture 12: Spectral sequences

2/15/15

1 Definition

A differential group \((E,d)\) (respectively algebra, module, vector space etc.) is a group (respectively algebra, module, vector space etc.) \(E\) together with a morphism \(d : E \rightarrow E\) such that \(d^2 = 0\). The homology \(H(E,d)\) of \((E,d)\) is defined to be the group

\[ H(E,d) = \text{Ker} \, d / \text{Image} \, d. \]

**Definition 1.1.** A spectral sequence is a sequence of differential groups \((E_n,d_n)\) for \(n \geq 2\) (or 1 or 0) such that \(E_n \cong H(E_{n-1},d_{n-1})\).

This definition is useful when \(E_2\) can be computed and the \(E_n\)'s stabilize to some \(E_\infty\) that we wish to compute.

2 Spectral sequence associated to an exact couple

An exact couple is a pair \(F,E\) of groups and a diagram

\[
\begin{array}{c}
F \\
\text{\|} \\
E \\
\downarrow^k \\
\text{\|} \\
F \\
\text{\|} \\
\end{array}
\]

which is exact. Define \(d = jk : E \rightarrow E\). Then, by exactness, \(d^2 = j(kj)k = 0\) because \(kj = 0\), i.e., \((E,d)\) is a differential group.
Given an exact couple, we can form another exact couple called the derived couple:

\[
\begin{array}{ccc}
F_1 & \xrightarrow{i_1} & F_1 \\
\downarrow{k_1} & & \downarrow{j_1} \\
E_1 & & \end{array}
\]

with the following definitions. \(F_1 = \text{Image}(i), \ E_1 = \text{H}(E, d), \ i_1 = i|_{F_1}, \ k_1 = k,\) and “\(j_1 = j_{i^{-1}}.\)” Since \(i\) is not invertible, the definition \(j_1 = j_{i^{-1}}\) really means that given \(f \in F_1 = i(F),\) we can choose \(f'\) such that \(if' = f.\) Then \(j_1 f = jf'.\)

In order for the definitions of \(j_1\) and \(k_1\) to make sense, we need to check that \(jf'\) does not depend on the choice of \(f',\) and \(k\) is a well-defined function on \(\text{Ker}(jk)/\text{Image}(jk)\) whose image is in the image of \(i.\) Note that \(jf'\) can be modified by \(j\) applied to an element in the kernel of \(i.\) This kernel is the image of \(k\) by construction. Thus, \(jf'\) can only be modified by an element in \(\text{Image}(jk),\) showing that \(j_1\) is well-defined. The others can also be checked in an entirely straight-forward manner.

**Exercise 2.1.** Show \((1)\) does indeed define an exact couple, i.e., the diagram is exact.

If you are not familiar with spectral sequences, it is a good idea to think a few of these things through.

Repeating this process, we obtain a spectral sequence \((E_n, d_n)\).

We can write down the terms of this spectral sequence explicitly. For example, \(F_n \subseteq F\) is the image of \(i^n\) and \(i_n\) is the restriction of \(i\) to \(F_n.\) \(d_n = j_n k_n\) is given by the formula “\(d_n = j_{i^{-n}} k.\)” Adams says that these explicit formulas probably come from a point of view due to Eilenberg, whereas the exact couple point of view is due to Massey. There is a short description of the explicit formula method in [L].

## 3 Convergence

A first approximation to what “convergence” means is that a spectral sequence converges to a group \(A\) if \(E_n = A\) for sufficiently large \(n.\) But this is not sufficiently general to be useful.

A group (respectively algebra, module, vector space etc.) \(A\) is said to be filtered by \(n \in \mathbb{Z}\) if there is a sequence of subgroups

\[
\ldots \subseteq \mathcal{F}_n A \subseteq \mathcal{F}_{n+1} A \subseteq \mathcal{F}_{n+2} A \subseteq \ldots
\]
of $A$. We will assume that $\cup_n F_n A = A$, and that $\cap_n F_n A = 0$.

A group (respectively algebra, module, vector space etc.) $A$ is said to be $\mathbb{Z}$-graded or just graded (respectively bigraded) if $A = \oplus_{n \in \mathbb{Z}} A_n$ (respectively $A = \oplus_{p,q \in \mathbb{Z}} A_{p,q}$). Call $A_p$ (resp. $A_{p,q}$) the $p$th (respectively $(p, q)$th) homogenous piece.

The associated graded of a filtered group $A$ is $\text{gr} A = \oplus_n F_{n+1} A / F_n A$.

Frequently spectral sequences have additional structure such that $(E_n, d_n)$ are graded (respectively bigraded). Suppose that $A$ is a filtered group (respectively filtered and graded). We say that $(E_n, d_n)$ converges to $A$ if for each $p$ (respectively $(p, q)$), the homogenous pieces of the $E_n$ stabilize to the associated graded of $A$. Note that when $A$ has a grading as well as a filtration, the associated graded is bigraded, so that’s consistent with having a bigrading on the spectral sequence. Degree considerations and indexing of spectral sequences can certainly be messy. On the other hand, spectral sequences turn out to be very powerful.

4 Spectral sequence associated to a double complex.

A double complex will be a bunch of groups (or modules etc.) $A^{p,q}$ for $\mathbb{Z} \ni p, q \geq 0$ and differentials $d : A^{p,q} \to A^{p-1,q}$, $d' : A^{p,q} \to A^{p-1,q-1}$ such that the differentials either commute, meaning $dd' = d'd$, or anti-commute, meaning $dd' + d'd = 0$. Both come up, and you can go from one to the other by changing $d'$ to $(-1)^p d'$. Let’s assume we are in the anti-commuting case.

Define $\text{Tot} A^{*,*}$ to be the differential graded group whose $n$th homogenous piece is $\oplus_{p+q=n} A^{p,q}$ and whose differential is $D = d + d'$. There are two natural spectral sequences associated to this double complex, and both converge to $H(\text{Tot} A^{*,*}, D)$. Here is one.

Note that there is a filtration $F_n = F_n \text{Tot} A^{*,*} = \oplus_{p \leq n} A^{p,q}$ on $\text{Tot} A^{*,*}$ such that $D : F_n \to F_n$. We have a short exact sequence of differential objects

$$0 \to F_n \to F_{n+1} \to F_{n+1}/F_n \to 0$$
which gives rise to a long exact sequence in homology with respect to \( D \)
\[
\ldots \to H_1 \mathcal{F}_n \to H_1 \mathcal{F}_{n+1} \to H_1 \mathcal{F}_{n+1}/\mathcal{F}_n \to H_{i-1} \mathcal{F}_n \to \ldots
\]  \( \tag{2} \)

Define
\[
F_1 = \oplus_{p,q} H_{p+q} \mathcal{F}_p
\]
\[
E_1 = \oplus_{p,q} H_{p+q} (\mathcal{F}_p/\mathcal{F}_{p-1}).
\]

Define \( i_1, j_1, \) and \( k_1 \) using the maps from the long exact sequence, meaning define \( i_1 : F \to F \) by the maps \( H_{p+q} \mathcal{F}_p \to H_{p+q} \mathcal{F}_{p+1} \) induced from the inclusions; define \( j_1 : F \to E \) by the maps \( H_{p+q} \mathcal{F}_{p+1} \to H_{p+q} \mathcal{F}_{p+1}/\mathcal{F}_p \) induced from the quotient; define \( k_1 : E \to F \) using the boundary maps. This is an exact couple because (2) is exact. Therefore we have constructed a spectral sequence.

The other spectral sequence is obtained by switching the roles of \( p \) and \( q \) in the definition of the filtration.

To use this spectral sequence and to see its convergence, it is useful to draw it. Place every \( A_{p,q} \) on the \((p, q)\) spot in the first quadrant. There are arrows \( d' \) forming vertical lines pointing down. Replace each \( A_{p,q} \) by \( A_{p,q} = \text{Ker} d'/\text{Image} d' \) the homology with respect to \( d' \). Note that \( D = d' \) on all \( \mathcal{F}_{n+1}/\mathcal{F}_n \). Thus \( E_1 = \oplus_{p,q} A_{p,q} \) is the direct sum of all the groups in our first quadrant. Moreover, the \( d_1 \) from our spectral sequence can be identified with \( d' \). Furthermore, the differential \( d_n \) is of bidegree \((-n, n-1)\). (Exercise.) In particular, for any \((p, q)\) eventually all of the differentials leaving or entering the group at the \((p, q)\)th spot are 0 so the spectral sequence converges. Call \( E_{\infty}(p,q) \) the limiting group in the \((p, q)\)th spot.

The filtration on \( \text{Tot} A^{*,*} \) induces a filtration on \( H_n \text{Tot} A^{*,*} \), by defining \( \mathcal{F}_p H_n \text{Tot} A^{*,*} \) to be the image of the map on \( H_n \) induced by \( \mathcal{F}_p \text{Tot} A^{*,*} \to \text{Tot} A^{*,*} \).

**Exercise 4.1.** Show that \( E_{\infty}^{(p,q)} = \mathcal{F}_p H_{p+q} \text{Tot} A^{*,*}/\mathcal{F}_{p-1} H_{p+q} \text{Tot} A^{*,*} \). In other words, the limiting groups can be identified with the associated graded of the homology of \( \text{Tot} A^{*,*} \).

**Example 4.2.** Given the commutative diagram with exact rows
\[
\begin{array}{ccc}
0 & \rightarrow & A \\
\downarrow f & & \downarrow g \\
0 & \rightarrow & A'
\end{array}
\quad
\begin{array}{ccc}
& & \rightarrow & C \\
& & \downarrow h \\
& & \rightarrow & C'
\end{array}
\quad
\begin{array}{ccc}
& & \rightarrow & 0
\end{array}
\]

it is a standard exercise to show that there is an exact sequence
\[
0 \rightarrow \text{Ker} f \rightarrow \text{Ker} g \rightarrow \text{Ker} h \rightarrow \text{coKer} f \rightarrow \text{coKer} g \rightarrow \text{coKer} h \rightarrow 0.
\]
This follows from the two spectral sequences just constructed. Namely, we can view this diagram as a double complex. Since the rows are exact, one of our spectral sequences converges to 0, so it follows that Tot has 0 homology. Thus the other spectral sequence must also converge to 0. This happens if and only if the claimed sequence is exact.

References


[L] Serge Lang, Algebra, Graduate Texts in Mathematics.


[V] Ravi Vakil, Course notes for Math 216.