Lecture 7: Cofiber sequences are fiber sequences

1/23/15

1 Cofiber sequences and the Puppe sequence

If \( f : X \to Y \) is a map of CW complexes, recall that the reduced mapping cone is the space \( Y \cup f\, CX = (Y \coprod X \wedge [0, 1]) / \sim \), where \((x, 1) \sim f(x)\). If we vary \( f \) by a homotopy, \( Y \cup f\, CX \) changes by a homotopy equivalence.

We may likewise form the reduced mapping cone of a map \( f : X \to Y \) in the stable homotopy category. \( f \) is represented by a function of spectra \( f : X' \to Y \), where \( X' \) is a cofinal subspectrum of \( X \). By replacing \( f \) by a homotopic map, we may assume that \( f'_n : X'_n \to Y_n \) is a cellular map. Then \( Y \cup f\, CX \) is the spectrum whose \( n \)th space is \( Y_n \cup f'_n\, CX'_n \) and whose structure maps are induced from those of \( X \) and \( Y \). This is well-defined up to isomorphism, because varying \( f \) by a homotopy does not change the isomorphism class of \( Y \cup f\, CX \).

If \( i : A \to X \) is the inclusion of a closed subspectrum, then define \( X/A \) be the spectrum whose \( n \)th space is \( X_n/A_n \) and whose structure maps are those induced from the structure maps of \( X \). The evident map \( X \cup i\, CA \to X/A \) is an isomorphism in the stable homotopy category because on the level of spaces we have homotopy equivalences which therefore induce isomorphism on \( \pi_* \).

**Definition 1.1.** A cofiber sequence is any sequence equivalent to a sequence of the form \( X \xrightarrow{f} Y \xrightarrow{i} Y \cup f\, CX \)

**Proposition 1.2.** ([A, III Prop 3.9]) For each \( Z \), the sequence \( [Y \cup f\, CX, Z] \to [Y, Z] \to [X, Z] \) is exact.

**Proof.** Since the composite \( X \to Y \cup f\, CX \) is null, we have that the image of \( [Y \cup f\, CX, Z] \to [Y, Z] \) is indeed in the kernel of \( [Y, Z] \to [X, Z] \). Suppose \( g : Y' \to Z \) is a function such that \( Y' \) is a cofinal subspectrum of \( Y \) and such that the associated morphism is null in \( [X, Z] \). We wish to construct a pmap \( Y \cup f\, CX \to Z \) extending the pmap \( g : Y \to Z \). To do this, choose a cofinal
subspecrum $X'$ of $X$ such that $gf$ is defined as a function on $X'$ and such that there is a function $H : X' \wedge [0,1]_+ \to Z$ giving a homotopy between $gf$ and the constant map. We checked that we may choose a cofinal subspectrum $Y''$ of $Y$ containing the image of $X'$. $H$ and $g$ determine a function $Y'' \cup_f CX' \to Z$. □

Any map can be extended to a cofiber sequence. In particular, we can extend cofiber sequences to the right

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX \to (Y \cup_f CX) \cup_i CY \to$$

Since $(Y \cup_f CX) \cup_i CY \cong \Sigma X$, we get that

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX \to \Sigma X \to \Sigma Y \to \Sigma (Y \cup_f CX) \to \Sigma \Sigma X \quad (1)$$

has all three term sequences cofiber sequences.

Note that desuspensions and suspensions of cofiber sequences are cofiber sequences. Applying $\Sigma^{-1}$ to the cofiber sequence $Y \cup_f CX \to \Sigma X \to \Sigma Y$, we have that $\Sigma^{-1} Y \cup_f CX \to X \to Y$ is a cofiber sequence. Thus we may continue (1) to the left.

**Corollary 1.3.** If $X \to Y \to Z$ is a cofiber sequence in the stable homotopy category, then for any $W$, the sequence

$$\ldots \to [X, W]_{n+1} \to [Z, W]_n \to [Y, W]_n \to [X, W]_n \to \ldots$$

is exact.

**Proof.** The sequence

$$\ldots \to [\Sigma^{n+1} X, W] \to [\Sigma^n Z, W] \to [\Sigma^n Y, W] \to [\Sigma^n X, W] \to \ldots$$

is exact by the above chain of cofiber sequences and Proposition 1.2. Since we have identified desuspension with a shift, suspension may also be identified with a shift. Thus $[\Sigma^n Y, W] = [Y, W]_n$. □

## 2 Fiber sequences are cofiber sequences

**Proposition 2.1.** If $X \xrightarrow{f} Y \xrightarrow{i} Z$ is a cofiber sequence in the stable homotopy category, then for any $W$, the sequence

$$\ldots \to [W, X]_n \to [W, Y]_n \to [W, Z]_n \to [W, X]_{n-1} \to \ldots$$

is exact.
Proof. As above, it suffices to show that

\[ [W, X] \to [W, Y] \to [W, Z] \]

is exact. Since the composite \( X \to Z \) is null, we have that the composite \( [W, X] \to [W, Z] \) is 0. Let \( g : W \to Y \) be a pmap such that \( ig \) is nullhomotopic. The choice of a null-homotopy gives the morphism \( h : CW \to Z \) from cone on \( W \) to \( Z \). We then obtain \( j \) and \( k \) in the commutative diagram

\[
\begin{array}{cccccc}
X & \xrightarrow{f} & Y & \xrightarrow{i} & Z & \xrightarrow{\Sigma f} & \Sigma Y \\
\downarrow{g} & & \downarrow{h} & & \downarrow{j} & & \downarrow{\Sigma g} \\
W & \xrightarrow{\omega} & W & \xrightarrow{\Sigma \omega} & CW & \xrightarrow{\Sigma \omega} & \Sigma W
\end{array}
\]

Since suspension is an equivalence, we have that the image of \( \Sigma^{-1} j \) under \( [W, X] \to [W, Y] \) is \( g \).

One could define a fiber sequence to be \( X \to Y \to Z \) such that the composite \( X \to Z \) was the constant map and such that the sequence satisfies the conclusion of Proposition 2.1. Dually, one could also define a cofiber sequence to be \( X \to Y \to Z \) such that \( X \to Z \) is null and satisfying the conclusion of 1.3. This is the same as the above (exercise: use a natural map and the 5 lemma to show it induces an isomorphism on \( \pi_\ast \)). Proposition 2.1 can therefore be stated by saying that fiber sequences and cofiber sequences are the same in the stable homotopy category.

References