Analysis I: Second Exam
February 27, 2006

Complete the FOUR problems below.
Complete Each Question on One Sheet of Paper, with Name on Each Sheet
Front Side of the Paper Only!
Give Complete and Clear Solutions! All problems are worth 20 points.

1. Let $f : [0,1] \longrightarrow \mathbb{R}$ be continuous. Show that f is *uniformly continuous*. That is, for all $\epsilon > 0$ there is a $\delta > 0$ so that for all $0 \leq x, y \leq 1$, if $|x-y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

Solution: Suppose that this is not the case. Then, for some $\epsilon > 0$ for all n there are points $|x_n - y_n| < 2^{-n}$, yet $|f(x_n) - f(y_n)| > \epsilon$. Now, $[0,1]$ is compact. Hence, there is a subsequence n_j such that x_{n_j} converges to some $0 < x < 1$. Then, y_{n_j} also converges to x. Continuity of f tells us that

$$f(x) = \lim_n f(x_n) = \lim_n f(y_n)$$

but the construction tells us $\liminf_n |f(x_n) - f(y_n)| > \epsilon$, which is a contradiction.

2. Define a function $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$F(x,y) = \begin{cases} x + y & x, y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

Find the points of continuity of F.

Solution: F is continuous at each point $(x, -x)$ for $x \in \mathbb{R}$. This is because $F(x,-x) = 0$. And for any sequence $(x_n, y_n) \longrightarrow (x, -x)$ along the values of the sequence for which both x_n and y_n are rational, we have $F(x_n, y_n) = x_n - y_n \longrightarrow 0$. Otherwise, the value of $F(x_n, y_n) = 0$.

3. Show that the continuous image of compact sets is again compact. Namely, if $K \subset \mathbb{R}^p$ is compact and $f : \mathbb{R}^p \longrightarrow \mathbb{R}^q$ is continuous, then $f(K) = \{f(x) : x \in K\}$ is compact.

Solution: Let $\mathcal{U}$ be an open cover for $f(K)$. As f is continuous, the inverse image of open sets is open. Hence, the collection of open set $f^{-1}(\mathcal{U}) = \{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of K. But then there are finitely many elements

$$f^{-1}(U_1),\, f^{-1}(U_2), \ldots f^{-1}(U_n)$$

that cover K. It follows that $U_1, \ldots, U_n$ cover $f(K)$. So we are done.

4. Suppose that $f_n : [-1,1] \longrightarrow \mathbb{R}$ are a sequence of functions which converge *uniformly* to a limiting function f. That is for all $\epsilon > 0$ there is a $N > 0$ so that for all $n \geq N$ and all $x \in [-1,1]$, $|f_n(x) - f(x)| < \epsilon$. Suppose that each f_n is continuous at $x = 0$. Show that f is continuous there.

Solution: Given $\epsilon > 0$ chose n so large that $|f_n(x) - f(x)| < \epsilon$ for all $x \in [-1,1]$. Then, choose $\delta > 0$ so that for all $|x| < \epsilon$ we have $|f_n(x) - f_n(0)| < \epsilon$. Then, it follows that

$$|f(x) - f(0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(0)| + |f_n(0) - f(0)| \leq 3\epsilon\,.$$

This does the job.